

The 2–category of species of dynamical patterns

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ABSTRACT. A new category \mathfrak{dp} , called of dynamical patterns addressing a primitive, non-geometrical concept of dynamics, is defined and employed to construct a 2-category $2 - \mathfrak{dp}$, where the irreducible plurality of species of context-depending dynamical patterns is organized. We propose a framework characterized by the following additional features. A collection of experimental settings is associated with any species, such that each one of them induces a collection of experimentally detectable trajectories. For any connector T , a morphism between species, any experimental setting E of its target species there exists a set such that with each of its elements s remains associated an experimental setting $T[E, s]$ of its source species, $T[\cdot, s]$ is called charge associated with T and s . The vertical composition of connectors is contravariantly represented in terms of charge composition. The horizontal composition of connectors and 2-cells of $2 - \mathfrak{dp}$ is represented in terms of charge transfer. A collection of trajectories induced by $T[E, s]$ corresponds to a collection of trajectories induced by E (equiformity principle). Context categories, species and connectors are organized respectively as 0, 1 and 2 cells of $2 - \mathfrak{dp}$ with factorizable functors via \mathfrak{dp} as 1-cells and as 2-cells, arranged themselves to form objects of categories, natural transformations between 1-cells obtained as horizontal composition of natural transformations between the corresponding factors. We operate a nonreductionistic interpretation positing that the physical reality holds the structure of $2 - \mathfrak{dp}$, where the fibered category \mathfrak{Cnt} of connectors is the only empirically knowable part. In particular each connector exists as an irreducible entity of the physical reality, and empirically detectable through the charges associated with it and experimentally represented by means of its equiformity principle. The algebraic structure of \mathfrak{Cnt} is experimentally detectable in terms of charge composition and charge transfer. \mathfrak{dp} widely generalizes the category of C^* -dynamical systems. The dynamical group is replaced by a \mathbf{top} -enriched category called dynamical category, the group action by the dynamical functor namely a functor of \mathbf{top} -enriched categories from the dynamical category to the category of unital topological $*$ -algebras naturally enriched over \mathbf{top} , finally an equivariant map between C^* -dynamical systems is replaced by a couple formed by a functor f between dynamical categories and a natural transformation from the composition of the dynamical functor of the source with f to the dynamical functor of the target. As an emblematic model we show that the equivariance under diffeomorphic actions of the flow of complete perfect fluids on general spacetimes is assembled into a species \mathbf{a} on the category \mathbf{St} of spacetimes and complete vector fields, with smooth maps relating the vector fields as morphisms. As a result the equivalence principle in general relativity emerges as the equiformity principle of the identity connector of \mathbf{a} . Said a quantum gravity a suitable species \mathbf{b} on \mathbf{St} such that the underlying topological $*$ -algebras are noncommutative, then the existence of a connector from \mathbf{a} to \mathbf{b} enables a quantum realization of the velocity of maximal integral curves of complete vector fields over spacetimes. When applied to Robertson-Walker spacetimes we establish that the Hubble parameter, the acceleration of the scale function and new constraints for its positivity evaluated on a subset of the range of the galactic time of a geodesic α , are expressed in terms of a quantum realization of the velocity of α . As a result the existence of a connector satisfying these constraints implies a positive acceleration and represents an alternative to the dark energy hypothesis.

Part 1

Species of dynamical patterns

1.1. Introduction

In order to establish when physical theories may be considered equivalent in all spacetimes, Fewster and Verch [13] define locally covariant theories and their embeddings in terms of the category of functors from the category of globally hyperbolic spacetimes \mathbf{Loc} to the abstract category of physical systems \mathbf{Phys} , for which the category \mathbf{CA}^* of C^* -algebras and $*$ -morphisms, represents a model. A similar concept in the special case of \mathbf{CA}^* was previously discussed in Brunetti, Fredenhagen and Verch [8], where in order to address in a general covariant setting the concept of quantum field, they defined any locally covariant theory as a functor from essentially \mathbf{Loc} to \mathbf{CA}^* . They regarded a quantum field as a natural transformation between functors obtained by composing locally covariant theories with the forgetful functor from \mathbf{CA}^* to the category of topological spaces. In [13] and [8] the target categories \mathbf{Phys} and \mathbf{CA}^* respectively are interpreted essentially as the collection of kinematical systems with embeddings as morphisms, allowing the dynamical transformations to be realized in terms of morphisms. In what follows the categories modeling \mathbf{Phys} are named kinematical categories.

We instead retain that dynamics are actualized by objects of a category, they in general are not byproducts of geometrical transformations, although they could be covariant under geometrical actions. Thus our initial posit which will be later extended, reads as follows.

Dynamics is a primitive collection of entities organized to form the category \mathfrak{dp} .

The meaning of primitive will be later formalized. \mathfrak{dp} is constructed in Def. 1.4.1 and Cor. 1.4.5 and called category of dynamical patterns. It is a nontrivial generalization of the category \mathfrak{ds} of topological dynamical systems with equivariant maps as morphisms, where a topological dynamical system is determined by a morphism in the category of topological groups whose target is the group of continuous $*$ -automorphisms of a unital topological $*$ -algebra endowed with the topology of simple convergence. Let us call dynamical group the source object of a dynamical system.

In constructing \mathfrak{dp} the dynamical group is replaced by a *dynamical category*, that is a category enriched¹ over \mathbf{top} , the category of topological spaces, and the group morphism replaced by the *dynamical functor*, namely a functor of \mathbf{top} -enriched categories from the dynamical category to the naturally \mathbf{top} -enriched category \mathbf{tsa} of unital topological $*$ -algebras and continuous $*$ -morphisms. Morphisms of \mathfrak{dp} are couples formed by a \mathbf{top} -functor f from the dynamical category of the target to the dynamical category of the source and a natural transformation from the composition of the dynamical functor of the source composed with f , to the dynamical functor of the target, generalizing the concept of equivariant morphisms in \mathfrak{ds} .

\mathbf{CA}^* is trivially embedded in \mathfrak{cds} , the subcategory of \mathfrak{ds} formed by C^* -dynamical systems, and \mathfrak{ds} is embedded in \mathfrak{dp} . The reasons to prefer in the definition of \mathfrak{dp} the category of topological $*$ -algebras rather than \mathbf{CA}^* or an abstract \mathbf{top} -enriched category reside in what follows. In the first case it is in order to embed not only \mathfrak{cds} but also

¹to be precise is a weaker version of the standard enrichment we call quasi-enrichment.

W^* -dynamical systems which are continuous w.r.t. the sigma-weak topology. In the second case it is in order to model physical theories, which require a framework to produce experimentally detectable values. Finally as we shall see below, the requisite to address in a compact and elegant way the geometric equivariance of the flow generated in particular by perfect fluids on spacetimes, forces us to move from dynamical groups to dynamical categories. The following properties characterize \mathfrak{dp} .

- (1) A species contextualized in a category, is a functor from this category to \mathfrak{dp}^2 , and it encodes a collection of context-depending *dynamical patterns* equivariant under action of the morphisms of the context category.
- (2) A collection of experimental settings $Exp(\mathfrak{a})$ is associated with any species \mathfrak{a} . Roughly an experimental setting is a couple (\mathfrak{S}, R) , where R maps any context M into a subcategory of the dynamical category of the dynamical pattern $\mathfrak{a}(M)$, while \mathfrak{S} maps M into a R_M -fibered family of continuous positive linear functionals over the topological $*$ -algebras underlying $\mathfrak{a}(M)$. Each functional stands for a statistical ensemble whose strength³ is the value the functional assumes at the identity. R and \mathfrak{S} are equivariant with respect to the geometrical action of the context category and most importantly \mathfrak{S} is equivariant under the action of the *dynamical subcategories represented by R via the conjugate of the dynamical functors* Def. 1.4.24. The fact that we consider dynamical subcategories reflects the eventual occurrence of broken dynamical symmetries.
- (3) To any species \mathfrak{a} , any context M and any couple of objects x, y of the dynamical category of $\mathfrak{a}(M)$ is assigned a family of experimentally detectable trajectories whose initial conditions are represented by couples of statistical ensembles and observables and such that the dynamics is realized by morphisms from x to y via the *dynamical functor* of $\mathfrak{a}(M)$ Def. 1.4.22. By restriction of the initial conditions a family of trajectories can be assigned to any experimental setting of \mathfrak{a} .
- (4) A connector is a natural transformation between species contextualized over the same category and it is decoded by the two diagrams exposed in Lemma 1.4.41, in which the category \mathfrak{dp} transparently determines the dynamical nature of the connector. We have the following properties
 - (a) **Charges.** Any connector T induces a set valued map $\Gamma(\cdot, T)$ over the set $Exp(c(T))$ of experimental settings of the target species of T and a function $T[\cdot, \cdot]$, mapping any couple (\mathfrak{Q}, s) where $\mathfrak{Q} \in Exp(c(T))$ and $s \in \Gamma(\mathfrak{Q}, T)$ into an experimental setting $T[\mathfrak{Q}, s] \in Exp(d(T))$ of the source species of T Thm.

²Exactly species are functors valued in the category $\mathfrak{Ch}\mathfrak{du}$, defined in Def. 1.4.18 and Cor. 1.4.19, however there exists a canonical functor from \mathfrak{dp} to $\mathfrak{Ch}\mathfrak{du}$ permitting to associate with any functor valued in \mathfrak{dp} a species. It is worthwhile remarking that the natural transformation between functors with values in \mathfrak{tsa} present in the definition of the morphisms of \mathfrak{dp} , is replaced in the $\mathfrak{Ch}\mathfrak{du}$ case with a natural transformation between functors with values in the category \mathfrak{ptsa} whose object set is as \mathfrak{tsa} but whose morphism set is the subset of linear positive continuous maps T between unital topological $*$ -algebras such that $T(1) \leq 1$. By dropping the request on T of being an algebra morphism will allow later to consider connectors between classical and quantum species, but at this stage of the discussion this point is irrelevant.

³strength with the meaning used in [11].

- 1.4.46(1). The map $T[\cdot, s]$ is called charge associated with T and s . If T connects species of dynamical systems the degeneration is removed and $T[\mathfrak{Q}]$ stands for $T[\mathfrak{Q}, s]$ [31].
- (b) **The vertical composition of connectors is contravariantly represented as charge composition.**
- (i) *general connectors*: under suitable hypothesis there exists a contravariant representation of the vertical composition of connectors in terms of composition of charges Cor. 1.4.48;
 - (ii) *connectors of species of dynamical systems*: if \mathfrak{a} is a species of dynamical systems the result is stronger, fixed a context category \mathfrak{D} , the assignments $\mathfrak{a} \mapsto \text{Exp}(\mathfrak{a})$ and $T \mapsto T[\cdot]$ determine a contravariant functor at values in \mathbf{set} and defined on the functor category of species contextualized on \mathfrak{D} with connectors as morphisms [31].
- (c) **The horizontal composition of connectors with 2-cells is represented as charge transfer.** For any connector T and any 2-cell L $*$ -composable to the right with T we have that $T * L$ is a connector, such that the charge $(T * L)[\cdot, r]$, for a suitable r depending by s , maps the pullback through y of any experimental setting \mathfrak{Q} of the target species of T into an experimental setting which is included in the pullback through x of the experimental setting obtained by mapping \mathfrak{Q} through the charge $T[\cdot, s]$. Here x and y are the source and target of L respectively, and s is a suitable element of $\Gamma(\mathfrak{Q}, T)$ Cor. 1.4.53.
- (d) **Equiformity principle**⁴. Let T be a connector from the species \mathfrak{a} to the species \mathfrak{b} , \mathfrak{Q} be an experimental setting of \mathfrak{b} and $s \in \Gamma(\mathfrak{Q}, T)$. Thus for all contexts M, N and morphisms $\phi : M \rightarrow N$ we have that the map obtained by conjugate action of $T_1^m(N)$ over any suitable trajectory of \mathfrak{a} , relative to N and assigned to $T[\mathfrak{Q}, s]$ equals the map obtained by conjugate action of $b_1^m(\phi)$ over a trajectory of \mathfrak{b} , relative to M and assigned to \mathfrak{Q} . Here $T_1^m(N)$ is the morphism map of a functor, determined by T , from the dynamical category of $\mathfrak{b}(N)$ to the dynamical category of $\mathfrak{a}(N)$ and $b_1^m(\phi)$ is the morphism map of a functor, determined by \mathfrak{b} and ϕ , from the dynamical category of $\mathfrak{b}(N)$ to the dynamical category of $\mathfrak{b}(M)$ Thm. 1.4.46(5).
- (5) Species provide a great variety of context-depending dynamics which cannot be modeled by employing functors valued in \mathbf{CA}^* or which would require ad hoc constructions involving morphisms in the context category.
- (a) The diffeomorphic equivariance of the flow generated by perfect fluids on spacetimes is the emblematic model of a species in \mathfrak{dp} Cor. 1.6.26, Thm. 1.6.24 and Thm. 1.6.12.

⁴The precise and general statement for links is established in Prp. 1.4.34 and physically interpreted in Prp. 1.4.35, while in Thm. 1.4.46 we show that any connector is a link.

- (b) The two symmetries of the relative Cauchy evolution established in [13, Prp. 3.7 and Prp. 3.8], are essentially two specific manifestations of the equiformity principle see Prp. 1.4.42 and the comment following it.

Context categories, species and connectors form 0,1 and 2 cells respectively of a 2-category $2 - \mathfrak{dp}$, such that the collection of all connectors can be organized in a fibered⁵ category \mathfrak{Cnt} over a subset of couples of species and be provided with a partial internal operation, the vertical composition \circ , and a module structure over the collection of 2-cells induced by the horizontal composition Prp. 1.5.14.

We regard $2 - \mathfrak{dp}$, and in particular \mathfrak{Cnt} , a *nontrivial dynamics-oriented* generalization of the category of covariant sectors in algebraic quantum field theory, and of the category of unitary net representations defined by Brunetti and Ruzzi in [9], We point out what follows.

- (1) New it is the construction of \mathfrak{dp} to model a dynamical pattern as a functor between **top**-enriched categories. In our framework dynamical phenomena in general reflect structural properties of primitive entities rather than be a byproduct of geometric transformations induced by morphisms in the context category.
- (2) New it is the structure of experimental setting. Our definition extends that of state space in [8], since it includes the dynamics and it extends the state space associated with a covariant sector in Doplicher, Haag and Roberts [12], since it extends the Poincaré action to a dynamical category action. More in general we introduce the concept of equivariance under action of dynamics of *non-geometrical* origin.
- (3) New it is the use of natural transformations and in particular connectors to construct charges and as a result state spaces which are covariant under dynamical action. Thus a connector extends the concept of covariant sector of [12] and its generalization in [9] to include dynamics of *non-geometrical* origin.
- (4) New it is the use of vertical composition of natural transformations and in particular of connectors, to generalize the concept of charge composition.
- (5) New it is the concept of charge transfer.
- (6) New it is the **equiformity principle** in particular for dynamics of *non-geometric* origin. As a natural transformation a connector is an *embedding* of species, in such a role is analog to a natural transformation between theories [13, 8]. Nevertheless a connector embeds functors valued in \mathfrak{dp} rather than in **Phys** or in any kinematical category. *Exactly because of the peculiar structure of \mathfrak{dp} which encodes directly the concept of **dynamics**, the connector encrypts in a natural way empirical information - concerning the correspondence between trajectories associated with its target species with those associated with its source species - decoded in terms of its equiformity principle. Now since \mathfrak{dp} permits to address dynamics which are*

⁵fibered here means simply a collection of categories labelled by some set.

not geometrically determined⁶, it results impossible to convey the above information by exploiting Phys or any kinematical category⁷. Moreover even if the dynamics is geometrically determined, the equiformity principle unifies diverse symmetries by explaining them as particular consequences. More explicitly the first commutative diagram in Lemma 1.4.41 integrates them as specific outcomes of the commutativity of subdiagrams, see Prp. 1.4.42 and the comment following it.

Our interpretation later referred as Ep, is as follows Posit 1.5.16.

The physical reality is intrinsically dynamical and it is structured by $2 - \mathfrak{dp}$, the only knowable part being \mathfrak{Cnt} the fibered category of connectors. Each connector is an irreducible entity empirically detectable by means of the charges associated with it and by employing its equiformity principle. The structure of \mathfrak{Cnt} as a whole is empirically detectable by the representation of the vertical composition in terms of charge composition and by the representation of the horizontal composition in terms of charge transfer.

We shall return later on the equiformity principle, here we have two remarks. Firstly let $1_{\mathfrak{dp}}$ be the identity species contextualized in \mathfrak{dp} , and $1_{1_{\mathfrak{dp}}}$ be the identity natural transformation whose source and target species equal $1_{\mathfrak{dp}}$, then $1_{1_{\mathfrak{dp}}}$ is a connector and it is primitive meaning that it is an identity with respect to the vertical and horizontal composition in \mathfrak{Cnt} . Therefore if we identify \mathfrak{dp} with the connector $1_{1_{\mathfrak{dp}}}$, then according to Ep, the category \mathfrak{dp} is an existing primitive entity, thus making precise the meaning primitive used above to characterize \mathfrak{dp} . Secondly the empirical representation of a species emerges in terms of the collection of the experimental settings generated by all its sectors and in terms of the equiformity principle of all its sectors, where a sector is a connector whose source equal the target.

As we announced the dynamical pattern approach appears to be required in order to encode with only one structure the equivariance under diffeomorphic actions of the flow of complete perfect fluids on spacetimes. To show this we briefly describe the construction of the functor \mathfrak{a} valued in \mathfrak{dp} and defined on the category \mathbf{St}_n . Roughly \mathbf{St}_n is the category of the couples (\mathcal{M}, U) , where \mathcal{M} is a n -dimensional spacetime and U is an observer field on \mathcal{M} , namely a complete timelike unit future-pointing smooth vector field on \mathcal{M} , with smooth maps between spacetimes preserving the orientation and relating the observer fields, as morphisms.

In particular with $n = 4$ the observer field U on \mathcal{M} can be the component of a perfect fluid (ρ, p, U) on \mathcal{M} , where the integral curves of U describe the trajectories of particles moving in the gravitational field described by the metric tensor of \mathcal{M} and subject to a density energy and density pression ρ and p respectively, [25, Def. 12.4]. As a result we have that if (\mathcal{M}, U) and (\mathcal{N}, V) are objects of \mathbf{St}_4 , such that U is the component of a perfect fluid on \mathcal{M} and there exists a diffeomorphism relating U and V , then V is the component of a perfect fluid on \mathcal{N} , Thm. 1.6.12

⁶given a functor \mathfrak{a} from \mathfrak{D} to $\mathfrak{Ch\ddot{u}v}$ and an object M of \mathfrak{D} we say that the dynamics of $\mathfrak{a}(M)$ is geometrically determined if the morphism map $\tau_{\mathfrak{a}(M)}$ of its dynamical functor factorizes through \mathfrak{a}_3 .

⁷more specifically it is impossible to obtain Lemma 1.4.41 if we replace $\mathfrak{Ch\ddot{u}v}$ with \mathbf{CA}^* or more in general with Phys.

As a first step we define the collection of \mathbf{vf} -topologies Def. 1.6.18 formed by functions ξ , mapping each object (\mathcal{M}, U) of \mathbf{St}_n into a relevant topology on $\mathcal{A}(M)$ making it a topological $*$ -algebra, where $\mathcal{A}(M)$ is the commutative $*$ -algebra of complex valued smooth maps on M the manifold supporting \mathcal{M} . Now fixed a \mathbf{vf} -topology ξ , the idea behind the construction of \mathfrak{a} it is to associate with any object (\mathcal{M}, U) of \mathbf{St}_n the dynamical pattern whose dynamical category denoted by $[M, U]$ holds as object set the collection of open subsets of \mathcal{M} , with morphisms the real numbers that via the flow of the complete vector field U map one open set into the other. The dynamical functor $F_{[M, U]}$ is such that its object map sends any open subset W of \mathcal{M} to $\mathcal{A}(W)$ provided by the topology inherited by the topology $\xi_{[M, U]}$, while its morphism map $F_{[M, U]}^m$ sends any real number t into the conjugate on $\mathcal{A}(W)$ of the flow of U evaluated in t . Finally for any morphism ϕ between two objects (\mathcal{M}, U) and (\mathcal{N}, V) the value in ϕ of the morphism map of \mathfrak{a} is the couple formed by a \mathbf{top} -functor f_ϕ from $[N, V]$ to $[M, U]$ and a natural transformation T_ϕ from $F_{[M, U]} \circ f_\phi$ to $F_{[N, V]}$ Thm. 1.6.24 and Cor. 1.6.26. The definition of \mathbf{vf} -topology is intrinsically related to \mathbf{St}_n and provides the minimal requirements in order to ensure the continuity of $F_{[M, U]}^m$ and T_ϕ .

In the remaining of this introduction we discuss interpretational features of the equiformity principle. Let us start by remarking from the above example that

- (1) The identity natural transformation of the functor \mathfrak{a} realizes the equivalence principle of general relativity in particular providing diffeomorphic covariance of the integral curves of complete perfect fluids Cor. 1.6.28.
- (2) Item (1) suggests to interpret the equiformity principle induced by any connector as a generalized equivalence principle between the source and target species.

In extreme synthesis we can say that the equiformity principle of \mathbf{T} roughly affirms that *a collection of trajectories assigned to any experimental setting \mathcal{Q} of the target species of \mathbf{T} corresponds to a collection of trajectories assigned to the experimental setting $\mathbf{T}[\mathcal{Q}, s]$ of the source species of \mathbf{T} for any $s \in \Gamma(\mathcal{Q}, \mathbf{T})$.*

Here what we point out is the correspondence between target and source species. Let us analyze some consequences of this principle when the source is a classical species and the target is a quantum species. Here by classical (quantum) species we mean a species such that it is commutative (noncommutative) the algebra associated with any context and any object of the dynamical category of the species. Then the principle establishes that classical and quantum trajectories correspond.

Notice that we are saying that *the dynamical evolution of classical observables when measured against classical statistical ensembles, equals the dynamical evolution of suitable quantum observables when measured against suitable quantum statistical ensembles*. Incidentally the main outcome of the reductionistic point of view is to regard general relativity a coarse grain approximation of, and then worthy to be reduced to, a theory where spacetime emerges from a more fundamental quantum entity, or at least where the gravitational field is quantized. Thus in both cases according to the reductionistic view, general relativity is compelled to be reduced and then replaced by a quantum theory of gravity.

Instead according Ep classical and quantum species coexist and this coexistence is empirically detectable in terms of the equiformity principle of the connectors between them.

More precisely assume that there exist b and T , where b is a (strict) quantum gravity, namely a suitable functor from the category St_n to $\mathfrak{D}\mathfrak{P}$ such that for any context (M, U) of St_n the dynamical category of $b(M, U)$ is $[M, U]$ and for any open set W of M the topological $*$ -algebra $\mathcal{A}_{b((M,U))}(W)$ is noncommutative, while T is a natural transformation from the species of general relativity a to the quantum gravity species b . Thus the equiformity principle of T establishes in particular the *quantum realization of the velocity of maximal integral curves of the complete vector field U* Cor. 1.6.40.

Clearly this sort of classical-quantum coexistence is automatically precluded by the actual reductionistic paradigm, however the equiformity principle produces experimentally testable equalities, that can be employed in order to opt for the paradigm embodied by Ep or for reductionism.

The paper is organized as follows. We start in section 1.3 by introducing the propensity map slightly generalizing the usual state-effect duality. In section 1.4 we introduce the category of dynamical patterns and the category of channels and devices. Define the concept of species, experimental settings of a species, links between species and the fundamental equiformity principle for links. We prove that any connector is a link between any experimental setting of its target and a suitable experimental setting of its source, thus providing an equiformity principle. Then we establish charge composition and charge transfer of connectors. These represent three of the five main results of the paper. In section 1.5 we introduce the general language to address the 2-category $2 - \mathfrak{D}\mathfrak{P}$ and the fibered category of connectors. In section 1.6.1 we construct in the fourth main result an example of species of dynamical patterns namely the n -dimensional classical gravity species a^n , and state that the equivalence principle of general relativity emerges as the equiformity principle of the connector associated with a^n . Finally in section 1.6.2 we define the collection of quantum gravity species and prove that the existence of a connector T from a^n and an n -dimensional strict quantum gravity species provides a quantum realization of the velocity of maximal integral curves of complete vector fields over spacetimes Cor. 1.6.40. As an application to Robertson-Walker spacetimes we establish that the Hubble parameter, the acceleration of the scale function and the new constraints of its positivity evaluated over a subset of the range of the galactic time of a geodesic α , are expressed in terms of a quantum realization of the velocity of α Thm. 1.6.41 and Cor. 1.6.43. *Therefore the positivity of the acceleration follows as a result of the existence of T satisfying these constraints rather than the existence of dark energy.*

1.2. Terminology and preliminaries

In the entire paper we consider the Zermelo-Fraenkel theory together the axiom of universes stating that for any set there exists a universe containing it as an element [17, p. 10]. See [1, Expose I Appendice] for the definition and properties of universes, see also [20, p. 22] and [3, §1.1].

1.2.0.1. *Sets.* Given two sets A, B let $\mathcal{P}(A)$ denote the power set of A , \hookrightarrow denotes the set embedding and for maps g, f such that $d(g) = c(f)$ and $B \subset d(f)$ we let $(g \circ f)(B)$ denote the image of B through the map $g \circ f \circ (B \hookrightarrow d(f))$. Let $\text{ev}_{(\cdot)}$ denote the evaluation map, i.e. if $F : A \rightarrow B$ is any map and $a \in A$, then $\text{ev}_a(F) := F(a)$. \mathbb{R} and \mathbb{C} are the fields of real and complex numbers provided by the standard topology, set $\mathbb{N}_0 := \mathbb{N} - \{0\}$ and $\mathbb{R}_0 := \mathbb{R} - \{0\}$, while $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ provided by the topology of one-point compactification. If A is any set then 1_A or simply 1 , is the identity map on A . For any semigroup S let S^{op} denote the opposite semigroup, while for any subset A of S let $\langle A \rangle$ denote the subsemigroup of S generated by A . If S, X are topological spaces, let $\mathcal{C}(S, X)$ denote the set of continuous maps on S and into X , and $Op(S)$, $Ch(S)$ and $Comp(S)$ denote the sets of open, closed and compact subsets of S , while $\mathcal{B}(S)$ the σ -field of Borel subsets of S . If $A \subset S$ then \overline{A} or $\text{Cl}(A)$ denotes the closure of A in S .

1.2.0.2. *Categories.* Let A, B and C be categories. Let $Obj(A)$ denote the set of the objects of A , often we let $x \in A$ denote $x \in Obj(A)$. For any $x, y \in A$ let $Mor_A(x, y)$ be the set of morphisms of A from x to y , also denoted by $A(x, y)$, let 1_x the unit morphism of x , while $Inv_A(x, y) = \{f \in Mor_A(x, y) \mid (\exists g \in Mor_A(y, x))(f \circ g = 1_y, g \circ f = 1_x)\}$ denotes the possibly empty set of invertible morphisms from x to y . Set $Mor_A = \bigcup \{Mor_A(x, y) \mid x, y \in A\}$ and $Inv_A = \bigcup \{Inv_A(x, y) \mid x, y \in A\}$, while $Aut_A(y) = Inv_A(y, y)$, for $y \in A$, and $Aut_A = \bigcup \{Aut_A(x) \mid x \in A\}$. For any $T \in Mor_A(x, y)$ we set $d(T) = x$ and $c(T) = y$, while the composition on Mor_A is always denoted by \circ . Let A^{op} denote the opposite category of A , [20, p. 33]. Let $\text{Fct}(A, B)$ denote the category of functors from A to B and natural transformations provided by pointwise composition, see [17, 1.3], [20, p.40], [3, p. 10]. Let us identify any $F \in \text{Fct}(A, B)$ with the couple (F_o, F_m) , where $F_o : Obj(A) \rightarrow Obj(B)$ said the object map of F , while $F_m : Mor_A \rightarrow Mor_B$ such that $F_m^{x,y} : Mor_A(x, y) \rightarrow Mor_B(F_o(x), F_o(y))$ where $F_m^{x,y} = F_m \upharpoonright Mor_A(x, y)$ for all $x, y \in A$. Often and only when there is no risk of confusion we adopt the standard convention to denote F_o and F_m simply by F . Let $1_A \in \text{Fct}(A, A)$ be the identity functor on A , defined in the obvious way. Let $\circ : \text{Fct}(B, C) \times \text{Fct}(A, B) \rightarrow \text{Fct}(A, C)$ denote the vertical composition of functors [17, Def. 1.2.10], [20, p. 14], for any $\sigma \in \text{Fct}(A, B)$, the identity morphism 1_σ of σ in the category $\text{Fct}(A, B)$ is such that $1_\sigma(M) = 1_{\sigma_o(M)}$, for all $M \in A$. Let $\beta * \alpha \in Mor_{\text{Fct}(A, C)}(H \circ F, K \circ G)$ be the Godement product or horizontal composition between the natural transformations β and α , where $H, K \in \text{Fct}(B, C)$ and $F, G \in \text{Fct}(A, B)$ while $\beta \in Mor_{\text{Fct}(B, C)}(H, K)$ and $\alpha \in Mor_{\text{Fct}(A, B)}(F, G)$, see [3, Prp. 1.3.4] (or [20, p. 42] where it is used the symbol \circ instead of $*$). The product $*$ is associative, in addition we have the following rule for all $\gamma \in Mor_{\text{Fct}(A, B)}(H, L)$, $\alpha \in Mor_{\text{Fct}(A, B)}(F, H)$, and $\delta \in Mor_{\text{Fct}(B, C)}(K, M)$, $\beta \in Mor_{\text{Fct}(B, C)}(G, K)$, see [3, Prp. 1.3.5]

$$(1) \quad (\delta * \gamma) \circ (\beta * \alpha) = (\delta \circ \beta) * (\gamma \circ \alpha).$$

We have

$$(2) \quad \beta * 1_F = \beta \circ F_o,$$

moreover $1_G * 1_F = 1_{G \circ F}$, and $1_F \circ 1_G = 1_F$. For all categories $\mathfrak{D}, \mathfrak{F}$, all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{F})$, all $T \in \prod_{O \in \mathfrak{D}} \text{Mor}_{\mathfrak{F}}(\mathbf{a}(O), \mathbf{b}(O))$ and $S \in \prod_{O \in \mathfrak{D}} \text{Mor}_{\mathfrak{F}}(\mathbf{b}(O), \mathbf{c}(O))$ we set $S \circ T \in \prod_{O \in \mathfrak{D}} \text{Mor}_{\mathfrak{F}}(\mathbf{a}(O), \mathbf{c}(O))$ such that $(S \circ T)(M) := S(M) \circ T(M)$ for all $M \in \mathfrak{D}$.

Let B be a category, for any $x \in B$ set $\Delta_x := \bigcup \{\text{Mor}_B(x, y) \mid y \in B\}$ and $\Gamma_x := \bigcup \{\text{Mor}_B(y, x) \mid y \in B\}$. Let $(\cdot)^+$ and $(\cdot)_\star$ be maps defined on Mor_B such that for any $T \in \text{Mor}_B$ we have $T^+ : \Delta_{c(T)} \rightarrow \Delta_{d(T)}$ $S \mapsto S \circ T$, and $T_\star : \Gamma_{d(T)} \rightarrow \Gamma_{c(T)}$ $S \mapsto T \circ S$, moreover let $(\cdot)^* := (\cdot)^+ \circ (\cdot)^{-1} \upharpoonright \text{Inv}_B$, thus

$$(3) \quad \begin{aligned} (T \circ J)^+ &= J^+ \circ T^+, \\ (T \circ J)_\star &= T_\star \circ J_\star, \\ (U \circ W)^* &= U^* \circ W^*, \end{aligned}$$

for any $T, J \in \text{Mor}_B$, such that $(T, J) \in \text{Dom}(\circ)$, and $U, W \in \text{Inv}_B$ such that $(U, W) \in \text{Dom}(\circ)$. Moreover whenever it does not make confusion let T^+ , T_\star and W^* denote also their restrictions.

Let \mathcal{V} be a universe, then following [17], a set is \mathcal{V} -small if it is isomorphic to a set belonging to \mathcal{V} , and a set is a \mathcal{V} -set if it belongs to \mathcal{V} . A \mathcal{V} -category A is a category such that $\text{Mor}_A(M, N)$ is \mathcal{V} -small for any $M, N \in A$, and A is \mathcal{V} -small if it is a \mathcal{V} -category and $\text{Obj}(A)$ is \mathcal{V} -small. Let us add the following definition: \mathcal{T} is a \mathcal{V} -type category if it is a \mathcal{V} -category such that $\text{Obj}(\mathcal{T}) \simeq B$ and $B \subseteq \mathcal{V}$. Clearly any \mathcal{V} -small category is a \mathcal{V} -type category since $\mathcal{V} \subset \mathcal{P}(\mathcal{V})$. Moreover if \mathcal{V}_0 is a universe such that $\mathcal{V} \in \mathcal{V}_0$, then $\mathcal{V} \subset \mathcal{V}_0$ since $\mathcal{V}_0 \subset \mathcal{P}(\mathcal{V}_0)$, hence any \mathcal{V} -type category is a \mathcal{V}_0 -small category.

PROPOSITION 1.2.1. *Let A, B be categories and \mathcal{V} a universe, thus*

(1)

$$\begin{aligned} \text{Obj}(\mathbf{Fct}(A, B)) &\subset \mathcal{P}(\text{Obj}(A) \times \text{Obj}(B)) \times \mathcal{P}(\text{Mor}_A \times \text{Mor}_B), \\ \text{Mor}_{\mathbf{Fct}(A, B)}(f, g) &\subset \prod_{x \in \text{Obj}(A)} \text{Mor}_B(f_o(x), g_o(x)), \forall f, g \in \text{Obj}(\mathbf{Fct}(A, B)); \end{aligned}$$

- (2) *if A is \mathcal{V} -small and B is a \mathcal{V} -category, thus $\mathbf{Fct}(A, B)$ is a \mathcal{V} -category;*
(3) *if A and B are \mathcal{V} -small categories then $\mathbf{Fct}(A, B)$ is a \mathcal{V} -small category.*

PROOF. St.(1) follows easily by the definitions. St.(2) follows since the second equality in st.(1) and [17, Def. 1.1.1.]. For any \mathcal{V} -small category C we have $\text{Mor}_C \in \mathcal{V}$ since [17, Def. 1.1.1.(v)], thus the st.(3) follows by st.(2), the first equality in st.(1), and [17, Def. 1.1.1.(viii,iv)]. \square

Let \mathcal{V} -set be the category of \mathcal{V} -sets, functions as morphisms with map composition, and \mathcal{V} -cat be the 2-category whose object set is the set of \mathcal{V} -small categories, and for any A, B \mathcal{V} -small categories $\text{Mor}_{\mathcal{V}\text{-cat}}(A, B)$ is the \mathcal{V} -small category $\mathbf{Fct}(A, B)$, see Prp. 1.2.1(3).

In the remaining of the paper we assume fixed three universes $\mathcal{U}, \mathcal{U}_0, \mathcal{U}_1$ such that $\mathcal{U} \in \mathcal{U}_0 \in \mathcal{U}_1$, the existence of \mathcal{U}_0 and then \mathcal{U}_1 fixed \mathcal{U} being ensured by the axiom of universes.

Let **set** and **cat** denote \mathcal{U} –**set** and \mathcal{U}_0 –**cat** respectively, while let **Cat** denote \mathcal{U}_1 –**cat**.

Thus **set** is a \mathcal{U} –category but it is **not** an object of \mathcal{U} –**cat**. However **set** is a subcategory of \mathcal{U}_0 –**set** and any \mathcal{U} –type category is an object of **cat**, note that **set** is an \mathcal{U} –type category so an object of **cat**. Next $\mathcal{U}_0 \subset \mathcal{U}_1$ hence **cat** is a 2–subcategory of **Cat**.

For any structure S whenever we say “the set of the S ’s”, we always mean the subset of those elements of \mathcal{U} satisfying the axioms of S . Therefore for what said, letting \mathcal{S} be the category whose object set is the set of the S ’s and $Mor_{\mathcal{S}}(a, b)$ is the set of the morphisms of the structure S from a to b , with $a, b \in Obj(\mathcal{S})$, then \mathcal{S} is an object of **cat**, and via the forgetful functor is equivalent to a subcategory $\tilde{\mathcal{S}}$ of **set**. Let \mathcal{S} be called the category of structure S , and let $\tilde{\mathcal{S}}$ denote the image of \mathcal{S} via the forgetful functor.

Let **top** be the category such that $Obj(\mathbf{top})$ is the set of topological spaces, $Mor_{\mathbf{top}}(X, Y) = \mathcal{C}(X, Y)$ for all $X, Y \in \mathbf{top}$, and map composition as morphism composition. Let **tg** be the category such that $Obj(\mathbf{tg})$ is the set of topological groups, $Mor_{\mathbf{tg}}(G, H)$ is the set of continuous group morphisms on G and into H , for all $G, H \in \mathbf{tg}$, and map composition as morphism composition.

In [19, Def. 1.3.2] is presented a definition of enrichment over a category, here we need a weaker definition. Let A be a subcategory of **set** and B a category. B is said to be A –quasi enriched if

- $Mor_B(x, y) \in A$ for all $x, y \in B$;
- for all $T \in Mor_B$ and $y \in B$;
 - $T^+ \upharpoonright Mor_B(c(T), y) \in Mor_A(Mor_B(c(T), y), Mor_B(d(T), y))$,
 - $T_\star \upharpoonright Mor_B(y, d(T)) \in Mor_A(Mor_B(y, d(T)), Mor_B(y, c(T)))$.

Notice that B is a \mathcal{U} –category and any subcategory of B is A –quasi enriched. Clearly if B is A –quasi enriched then B^{op} is A –quasi enriched, while if any category is A –enriched then it is A –quasi enriched. If B, C are A –quasi enriched define $Fct_A(B, C)$ the subset of the $F \in Fct(B, C)$ such that for all $u, v \in B$

$$F_m^{u,v} \in Mor_A(Mor_B(u, v), Mor_C(F_o(u), F_o(v))).$$

If D is A –quasi enriched, then $G \circ F \in Fct_A(B, D)$ for any $F \in Fct_A(B, C)$ and $G \in Fct_A(C, D)$, where \circ is the vertical composition of functors. For any structure S we convey that \mathcal{S} –quasi enriched means $\tilde{\mathcal{S}}$ –quasi enriched.

1.2.0.3. Preordered topological linear spaces. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and X, Y be topological linear spaces over \mathbb{K} (\mathbb{K} –t.l.s. often simply t.l.s. if $\mathbb{K} = \mathbb{C}$). Let $\mathcal{L}(X, Y)$ denote the \mathbb{K} –linear space of continuous \mathbb{K} –linear maps from X to Y , set $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $X^* := \mathcal{L}(X, \mathbb{K})$. $\mathcal{L}_s(X, Y)$ is the \mathbb{K} –t.l.s. whose underlying linear space is $\mathcal{L}(X, Y)$ provided by the topology of pointwise convergence, while $\mathcal{L}_w(X, Y)$ is the locally convex linear space whose underlying linear space is $\mathcal{L}(X, Y)$ provided by the topology generated by the following set of seminorms $\{q_{(\phi, x)} \mid (\phi, x) \in Y^* \times X\}$, where $q_{(\phi, x)}(A) \doteq |\phi(Ax)|$. There exists a unique category \mathbb{K} –tls (or simply tls if $\mathbb{K} = \mathbb{C}$) whose object set is the set of all \mathbb{K} –t.l.s.’s, $Mor_{\mathbf{tls}}(X, Y) = \mathcal{L}(X, Y)$ for all $X, Y \in \mathbf{tls}$, and map composition as morphism composition. $Aut_{\mathbb{K}\text{–tls}}(X)$ will be provided with the topology induced by the one in $\mathcal{L}_s(X)$

$$(4) \quad (\cdot)^* \in Mor_{\mathbf{tg}}(Aut_{\mathbb{K}\text{–tls}}(X), Aut_{\mathbb{K}\text{–tls}}(X_s^*)).$$

If A, B have richer structure than that of t.l.s., then $\mathcal{L}(A, B)$ is for $\mathcal{L}(X, Y)$ where X, Y is the underlying t.l.s. underlying A, B respectively, similarly for A^* and $\mathcal{L}(A)$. In case X is a normed space we assume $\mathcal{L}(X)$ to be provided by the topology generated by the usual sup-norm. If X is any structure including as a substructure the one of normed space say X_0 , for example the normed space underlying any normed algebra, we let $\mathcal{L}(X)$ denote the normed space $\mathcal{L}(X_0)$. If X, Y are Hilbert spaces and $U \in \mathcal{L}(X, Y)$ is unitary then $\text{ad}(U) \in \mathcal{L}(\mathcal{L}(X), \mathcal{L}(Y))$ denotes the isometry defined by $\text{ad}(U)(a) := UaU^{-1}$, for all $a \in \mathcal{L}(X)$.

A preordered topological linear space (p.t.l.s) is an object X of \mathbb{R} -tls with a preorder, i.e. a reflexive, transitive relation \geq providing X with the structure of preordered linear space, see for the definition [5, II.15], and such that the set $X^+ := \{x \in X \mid x \geq 0\}$ of positive elements of X is closed. $a \leq b$ is for $b \geq a$ while $x > y$ is for $x \geq y$ and $x \neq y$, similarly for $x < y$. Let $[a, b] := \{x \in X \mid a \leq x \leq b\}$ and $(a, b) := \{x \in X \mid a < x < b\}$, for any $a, b \in X$. Any function from X to Y mapping X^+ into Y^+ is called positive. If X is any \mathbb{R} -linear space and C is a pointed convex cone in X , i.e. $0 \in C, C + C \subseteq C$ and $\lambda \cdot C \subseteq C$ for all $\lambda > 0$, then the relation $x \geq y$ iff $x - y \in C$ provides X by the structure of preordered linear space, see [5, II.12, Prp. 13] or [29, p. 20] (convex cone called wedge and preordered vector space called ordered vector space in [29]). For any two p.t.l.s.'s X, Y , define

$$(5) \quad \mathbf{P}(X, Y) := \{T \in \mathcal{L}(X, Y) \mid T(X^+) \subseteq Y^+\},$$

and set $\mathbf{P}(X) := \mathbf{P}(X, X)$ and $\mathfrak{P}_X \doteq \mathbf{P}(X, \mathbb{R})$. Note that $T \in \mathbf{P}(X, Y)$ is an order morphism since it is linear and $a \geq b$ iff $a - b \geq 0$. $\mathbf{P}(X, Y)$ is a pointed convex cone and it is closed in $\mathcal{L}_s(X, Y)$ since Y^+ is closed, hence the relation $T \geq S \Leftrightarrow T - S \in \mathbf{P}(X, Y)$ provides $\mathcal{L}_s(X, Y)$ with the structure of p.t.l.s. said canonical. Note that $T \geq S$ iff $T(x) \geq S(x)$, for all $x \in X^+$ and $S, T \in \mathcal{L}(X, Y)$. By construction $\mathcal{L}_s(X, Y)^+ = \mathbf{P}(X, Y)$, in particular $(X_s^*)^+ = \mathfrak{P}_X = \mathbf{P}(X, \mathbb{R}) = \{\phi \in X^* \mid \phi(X^+) \subseteq \mathbb{R}^+\}$. Let $\mathbf{K}(X) := \{T \in \mathcal{L}(X) \mid 0 \leq T \leq 1\}$. There exists a unique category ptls whose object set is the set of all the p.t.l.s.'s, $\text{Mor}_{\text{ptls}}(X, Y) = \mathbf{P}(X, Y)$ for all $X, Y \in \text{ptls}$ and map composition as morphism composition. $\text{Aut}_{\text{ptls}}(X)$ will be provided by the topology induced by the one in $\mathcal{L}_s(X)$. Let tls_{\geq} denote the full subcategory of \mathbb{R} -tls whose object set equals $\text{Obj}(\text{ptls})$.

1.2.0.4. Topological $*$ -algebras and their order structures. Here as a topological algebra (t.a.) we mean a complex algebra with a topology of Hausdorff providing it an object of tls, and such that the product is separatedly continuous. For any (unital) t.a. \mathcal{B} the semigroup \mathcal{B} has to be understood as the (unital) multiplicative semigroup underlying \mathcal{B} . As a topological $*$ -algebra ($*$ -t.a.) we mean a unital involutive algebra with a topology providing it a t.a. and w.r.t. which the involution is a continuous map. Let 1 denote by abuse of language the unit of any unital algebra. If X is in tls then $\mathcal{L}_s(X)$ is a unital t.a. provided by the composition of maps as the product. For any two $*$ -t.a. \mathcal{A}, \mathcal{B} define $\text{Hom}^*(\mathcal{A}, \mathcal{B})$ to be the set of the $\tau \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ such that $\tau(1) = 1$, $\tau(ab) = \tau(a)\tau(b)$ and $\tau(a^*) = \tau(a)^*$ for all $a, b \in \mathcal{A}$. There exists a unique category tsa such that $\text{Obj}(\text{tsa})$ is the set of the topological $*$ -algebras, $\text{Mor}_{\text{tsa}}(\mathcal{A}, \mathcal{B}) = \text{Hom}^*(\mathcal{A}, \mathcal{B})$ for all $\mathcal{A}, \mathcal{B} \in \text{tsa}$, and map composition as morphism composition. $\text{Aut}_{\text{tsa}}(\mathcal{A})$ will be provided by the topology induced by the one in $\mathcal{L}_s(\mathcal{A})$. Notice that according the general convention used here for

categories, for all $\mathcal{A} \in \mathbf{tsa}$ the symbol $1_{\mathcal{A}}$ denotes the identity morphism of the object \mathcal{A} , i.e. the identity map from \mathcal{A} to itself. This is the reason instead to denote simply by un the unit of \mathcal{A} .

For any subset S of \mathcal{B} set $S' := \{a \in \mathcal{B} \mid (\forall b \in S)(ab = ba)\}$ and $S'' := (S')'$ said the commutant and bicommutant of S respectively. Let \mathcal{A} be a topological $*$ -algebra. Define $\mathcal{A}_{ob} := \{a \in \mathcal{A} \mid a = a^*\}$, $\text{Pr}(\mathcal{A}) := \{p \in \mathcal{A}_{ob} \mid pp = p\}$, $\mathcal{A}^* := \{a^*a \mid a \in \mathcal{A}\}$, $\mathcal{A}^{\natural} := \{\sum_{k=0}^n a_k^* a_k \mid a \in \mathcal{A}^n, n \in \mathbb{N}\}$ and the set of positive elements of \mathcal{A}

$$\mathcal{A}^+ := \text{Cl}(\mathcal{A}^{\natural}).$$

Our definition of positive elements and therefore of partial order in \mathcal{A} differs from the one in [29] where instead it is used the set \mathcal{A}^{\natural} , and differs from that in [14]. However the three definitions of partial order in \mathcal{A} agree in case \mathcal{A} is a locally C^* -algebra, see below. The reason to choose such a set of positive elements, resides in the fact that we want to have the sets $\mathfrak{P}_{\mathcal{A}}$, $\mathfrak{E}_{\mathcal{A}}$ closed in \mathcal{A}_s^* , and the set $Ef(\mathcal{A})$ (if the product is jointly continuous also $\Theta_{\mathcal{A}}$) closed in \mathcal{A} , see later.

$0, 1 \in \mathcal{A}^+$ since $1*1 = 1$. \mathcal{A}_{ob} is an \mathbb{R} -linear space and it is closed in \mathcal{A} the involution being continuous, hence $\mathcal{A}^+ \subset \mathcal{A}_{ob}$ and clearly it is the closure of \mathcal{A}^{\natural} in the \mathbb{R} -t.l.s. \mathcal{A}_{ob} , moreover \mathcal{A}^{\natural} is a pointed convex cone of \mathcal{A}_{ob} thus \mathcal{A}^+ is a pointed convex cone since [5, II.13, Prp. 14]. Hence \mathcal{A}^+ is a pointed closed convex cone in the \mathbb{R} -t.l.s. \mathcal{A}_{ob} , therefore by defining $a \geq b$ iff $a, b \in \mathcal{A}_{ob}$ and $a - b \in \mathcal{A}^+$ we provide \mathcal{A}_{ob} with the structure of p.t.l.s. Now let \geq be called standard and denote the relation in \mathcal{A} inherited by \geq on \mathcal{A}_{ob} , and for any $\mathcal{B} \in \mathbf{tsa}$ by abuse of language set

$$(6) \quad \mathbf{P}(\mathcal{A}, \mathcal{B}) := \{T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \mid T(\mathcal{A}^*) \subseteq \mathcal{B}^+, T(\mathcal{A}_{ob}) \subseteq \mathcal{B}_{ob}\}.$$

Set $\mathbf{P}(\mathcal{A}) := \mathbf{P}(\mathcal{A}, \mathcal{A})$, $\mathfrak{P}_{\mathcal{A}} := \mathbf{P}(\mathcal{A}, \mathbb{C})$, $\mathfrak{P}_{\mathcal{A}}^{\natural} := \{\phi \in \mathfrak{P}_{\mathcal{A}} \mid \phi(1) \neq 0\}$, and $\mathfrak{E}_{\mathcal{A}} := \{\omega \in \mathfrak{P}_{\mathcal{A}} \mid \omega(1) = 1\}$. $\mathfrak{E}_{\mathcal{A}}$ is a closed convex set of \mathcal{A}_s^* , and with any $\phi \in \mathfrak{P}_{\mathcal{A}}^{\natural}$ one associates the continuous state $\frac{1}{\phi(1)}\phi$. We have

$$(7) \quad \mathbf{P}(\mathcal{A}, \mathcal{B}) = \{T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \mid T(\mathcal{A}^+) \subseteq \mathcal{B}^+, T(\mathcal{A}_{ob}) \subseteq \mathcal{B}_{ob}\},$$

since linearity and continuity, moreover

$$(8) \quad (\forall T \in \mathcal{L}(\mathcal{A}, \mathcal{B})) (T \in \mathbf{P}(\mathcal{A}, \mathcal{B}) \Leftrightarrow T \upharpoonright \mathcal{A}_{ob} \in \mathbf{P}(\mathcal{A}_{ob}, \mathcal{B}_{ob})).$$

Any element in $\mathbf{P}(\mathcal{A}, \mathcal{B})$ is an order morphism and

$$(9) \quad \text{Mor}_{\mathbf{tsa}}(\mathcal{A}, \mathcal{B}) \subset \mathbf{P}(\mathcal{A}, \mathcal{B}),$$

so the maps $\text{Obj}(\mathbf{tsa}) \ni \mathcal{A} \mapsto \mathcal{A}_{ob}$ and $\text{Mor}_{\mathbf{tsa}}(\mathcal{A}, \mathcal{B}) \ni T \mapsto T \upharpoonright \mathcal{A}_{ob}$ determine a functor from \mathbf{tsa} to \mathbf{ptls} . $\mathbf{P}(\mathcal{A}, \mathcal{B})$ is a pointed convex cone and it is closed in $\mathcal{L}_s(\mathcal{A}, \mathcal{B})$ since \mathcal{B}^+ is closed, so closed also in the \mathbb{R} -t.l.s. $\mathcal{L}_s(\mathcal{A}, \mathcal{B})_{\mathbb{R}}$ underlying $\mathcal{L}_s(\mathcal{A}, \mathcal{B})$. Therefore the relation $T \geq S \Leftrightarrow T - S \in \mathbf{P}(\mathcal{A}, \mathcal{B})$ provides $\mathcal{L}_s(\mathcal{A}, \mathcal{B})_{\mathbb{R}}$ with the structure of p.t.l.s. said canonical. Let us conven to denote the p.t.l.s. $\mathcal{L}_s(\mathcal{A}, \mathcal{B})_{\mathbb{R}}$ by $\mathcal{L}_s(\mathcal{A}, \mathcal{B})$ or simply $\mathcal{L}(\mathcal{A}, \mathcal{B})$, and to denote the p.t.l.s. $(\mathcal{A}_s^*)_{\mathbb{R}}$ by \mathcal{A}_s^* or simply \mathcal{A}^* . Let $\mathbf{K}(\mathcal{A}) := \{T \in \mathcal{L}(\mathcal{A}) \mid 0 \leq T \leq 1\}$. Thus by construction $\mathcal{L}_s(\mathcal{A}, \mathcal{B})^+ = \mathbf{P}(\mathcal{A}, \mathcal{B})$, in particular

$$(10) \quad (\mathcal{A}_s^*)^+ = \mathfrak{P}_{\mathcal{A}} = \mathbf{P}(\mathcal{A}, \mathbb{C}).$$

Let $\mathbf{P}_{\mathcal{A}}$ be the set of the linear functionals ϕ on \mathcal{A} such that $\phi(\mathcal{A}^+) \subseteq \mathbb{R}^+$, to do not be confused with the set $\mathbf{P}(\mathcal{A})$. Let $\mathbf{E}_{\mathcal{A}}$ be called the set of states of \mathcal{A} and defined as the subset of the $\psi \in \mathbf{P}_{\mathcal{A}}$ such that $\psi(1) = 1$. A functional in \mathcal{A} is hermitian if $\phi = \phi^*$, where $\phi^*(a) := \overline{\phi(a^*)}$ for all $a \in \mathcal{A}$. If $\phi \in \mathbf{P}_{\mathcal{A}}$ (respectively $\phi \in \mathbf{E}_{\mathcal{A}}$) then ϕ is positive (respectively a state) w.r.t. the definition in [29] and [14] i.e. $\phi(\mathcal{A}^*) \subseteq \mathbb{R}^+$, hence ϕ is hermitian since [14, Lemma 12.3], in particular $\phi(\mathcal{A}_{ob}) \subseteq \mathbb{R}$, therefore by (7)

$$(11) \quad \mathfrak{P}_{\mathcal{A}} = \{\phi \in \mathcal{A}^* \mid (\forall a \in \mathcal{A})(\phi(a^*a) \geq 0)\},$$

and so $\mathfrak{P}_{\mathcal{A}} = \mathbf{P}_{\mathcal{A}} \cap \mathcal{C}(\mathcal{A}, \mathbb{C})$ and $\mathfrak{E}_{\mathcal{A}} = \mathbf{E}_{\mathcal{A}} \cap \mathcal{C}(\mathcal{A}, \mathbb{C})$. Notice that since [14] defines to be positive a linear functional in \mathcal{A} if it maps \mathcal{A}^* into \mathbb{R}^+ , then by (11) the set $\mathfrak{P}_{\mathcal{A}}$ (respectively $\mathfrak{E}_{\mathcal{A}}$) is the set of continuous positive linear functionals (respectively continuous states) w.r.t. the definition in [14], hence our order structure in \mathcal{A}^* coincides with the one in [14]. If \mathcal{A} is a locally C^* -algebra \mathcal{A}^* is a closed cone since [14, Cor. 10.16, Thm. 10.17], then $\mathcal{A}^* = \mathcal{A}^\natural = \mathcal{A}^+$ and our order structure in \mathcal{A} equals those in [29] and [14]. If \mathcal{A} is a C^* -algebra then $\mathbf{P}_{\mathcal{A}} = \mathfrak{P}_{\mathcal{A}}$ and $\mathbf{E}_{\mathcal{A}} = \mathfrak{E}_{\mathcal{A}}$.

Let $Ef(\mathcal{A}) := \{a \in \mathcal{A} \mid 0 \leq a \leq 1\}$ the set of effects of \mathcal{A} , it is closed since \mathcal{A}^+ it is so, moreover $\mathbf{Pr}(\mathcal{A}) \subset Ef(\mathcal{A})$ indeed $1 - p \in \mathbf{Pr}(\mathcal{A})$ and clearly $\mathbf{Pr}(\mathcal{A}) \subset \mathcal{A}^+$. $\psi(e) \in [0, \psi(1)]$ for any $\psi \in \mathbf{P}_{\mathcal{A}}$ and $e \in Ef(\mathcal{A})$ since ψ is an order morphism. We can define in $Ef(\mathcal{A})$ a partial sum as the restriction of the sum in \mathcal{A} in the set of the $(a, b) \in Ef(\mathcal{A}) \times Ef(\mathcal{A})$ such that $a + b \in Ef(\mathcal{A})$. The domain of the partial sum is not empty since it contains the set of the $(p, q) \in \mathbf{Pr}(\mathcal{A})$ such that $pq = 0$, indeed $p + q \in \mathbf{Pr}(\mathcal{A})$. Define $\delta_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}^{\mathcal{A}}, a \mapsto (b \mapsto a^*ba)$, then $\delta_{\mathcal{A}}$ or simply δ is a semigroup morphism from the opposite semigroup \mathcal{A}^{op} of \mathcal{A} to the semigroup of linear continuous $*$ -preserving maps on \mathcal{A} , in addition it is continuous if the product in \mathcal{A} is jointly continuous. Moreover for all $a \in \mathcal{A}$ we have $\delta(a)(\mathcal{A}^+) \subseteq \mathcal{A}^+$, since $\delta(a)(\mathcal{A}^\natural) \subseteq \mathcal{A}^\natural$ and $\delta(a)$ is continuous, hence

$$(12) \quad \delta_{\mathcal{A}} : \mathcal{A}^{op} \rightarrow \mathbf{P}(\mathcal{A}) \text{ semigroup morphism.}$$

Note that $\delta(a)(xy) \neq \delta(a)(x)\delta(y)$ unless $a \in \mathbf{U}(\mathcal{A})$ the unitary group of \mathcal{A} , and that $\delta(a + b)(d) \neq \delta(a)(d) + \delta(b)(d)$, unless $a, b \in \{d\}'$ and $a^*b = 0$. This last inequality is at the basis of the interference phenomenon in quantum physics and it is clearly related to the noncommutative nature of the algebra \mathcal{A} .

If \mathcal{S} is any structure richer than the structure of topological spaces, then \mathcal{S} is a **top**-quasi enriched category providing $Mor_{\mathcal{S}}(x, y)$ by the topology of pointwise convergence for all $x, y \in \mathcal{S}$. Note that any of the following categories **tg**, **ℓℓs**, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, **ptℓs**, **ℓℓs_≥** and **tℓa** are examples of such a \mathcal{S} . If \mathbf{B} is a **top**-quasi enriched category, then $Aut_{\mathbf{B}}(x)$ is an eventually empty group whose product is separatedly continuous for all $x \in \mathbf{B}$. If \mathbf{C} is a **top**-enriched category, then $Aut_{\mathbf{C}}(x)$ is a topological group for all $x \in \mathbf{C}$ such that the inversion map in $Aut_{\mathbf{C}}(x)$ is continuous. A topological groupoid is a **top**-enriched groupoid (i.e. a category in which all the morphisms are invertible) with continuous inversion, so a topological group is (identifiable with) a **top**-enriched groupoid with a unique object whose inversion map is continuous.

1.3. Measures of observables and propensity map

Our interpretation, described in Def. 1.3.8, is based on the concept of propensity rather than probability, term here used with a different meaning with respect to the one ascribed to it in [32]. We retain propensity a primitive, measuring independent, structural property of any triplet formed by a transformation of statistical ensembles J , a statistical ensemble of samples ω and an effect e , and that this property admits an empirical representation in terms of frequency: it is the ratio to the samples of ω of the cases in which the effect e has been observed when tested over the samples of $J(\omega)$ Def. 1.3.4. The propensity map slightly generalizes the usual state-effect duality, since it allows to calculate the frequency over possibly bigger statistical ensembles Rmk. 1.3.11. Let us start with the following easy to prove

LEMMA 1.3.1. \mathbf{tls}_{\geq} is \mathbf{ptls} -quasi enriched, moreover the maps $X \mapsto X_s^*$ and $T \mapsto T^+$, where X_s^* is provided by the canonical structure of p.t.l.s., determines an element in $\mathbf{Fct}_{\mathbf{ptls}}(\mathbf{tls}_{\geq}^{op}, \mathbf{tls}_{\geq})$ such that $K(X)^+ \subseteq K(X_s^*)$, for all $X \in \mathbf{ptls}$. In other words we have for all $X, Y, Z \in \mathbf{ptls}$, $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X, Y)$

- (1) $\dagger \in \mathcal{L}_s(\mathcal{L}_s(X, Y), \mathcal{L}_s(Y_s^*, X_s^*))$;
- (2) $(S \circ T)^+ = T^+ \circ S^+$;
- (3) $P(X, Y)^+ \subseteq P(Y_s^*, X_s^*)$, i.e. $U^+(\mathfrak{P}_Y) \subseteq \mathfrak{P}_X$, for all $U \in P(X, Y)$;
- (4) $K(X)^+ \subseteq K(X_s^*)$.

REMARK 1.3.2. Lemma 1.3.1(1)&(2) generalize to the cases $X, Y \in \mathbf{tls}$, while Lemma 1.3.1(3)&(4) remains true if $X, Y \in \mathbf{tsa}$.

Next we define $\mathbf{Q}(\mathcal{A}, \mathcal{B})$ with a twofold role, inducing via conjugation suitable statistical ensemble transformations, see comments after Def. 1.3.4 and Prp. 1.3.7(3), and inducing effect transformations Prp. 1.3.5(3).

DEFINITION 1.3.3. Let $\mathcal{A}, \mathcal{B} \in \mathbf{tsa}$ define

$$\begin{aligned} \mathbf{Q}(\mathcal{A}, \mathcal{B}) &:= \{T \in P(\mathcal{A}, \mathcal{B}) \mid T(1) \leq 1\}, \\ \mathfrak{Q}(\mathcal{A}, \mathcal{B}) &:= \{T \upharpoonright Ef(\mathcal{A}) \mid T \in \mathbf{Q}(\mathcal{A}, \mathcal{B})\}, \end{aligned}$$

set $\mathbf{Q}(\mathcal{A}) := \mathbf{Q}(\mathcal{A}, \mathcal{A})$ and $\mathfrak{Q}(\mathcal{A}) := \mathfrak{Q}(\mathcal{A}, \mathcal{A})$. Define $\Theta_{\mathcal{A}} := \{a \in \mathcal{A} \mid a^*a \leq 1\}$ and set $\gamma_{\mathcal{A}} := \delta_{\mathcal{A}} \upharpoonright \Theta_{\mathcal{A}}$, moreover set $\delta_{\mathcal{A}}^+ := (\dagger) \circ \delta_{\mathcal{A}}$, $\gamma_{\mathcal{A}}^+ := (\dagger) \circ \gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{A}, e} : \Theta_{\mathcal{A}} \ni a \mapsto \gamma_{\mathcal{A}}(a) \upharpoonright Ef(\mathcal{A})$.

We have $(\cdot)^* \in \mathbf{Mor}_{\mathbf{tg}}(\mathbf{Aut}_{\mathbf{ptls}}(X), \mathbf{Aut}_{\mathbf{ptls}}(X_s^*))$ for any $X \in \mathbf{ptls}$, since Lemma 1.3.1(3)&(1), so $(\cdot)^* \in \mathbf{Mor}_{\mathbf{tg}}(\mathbf{Aut}_{\mathbf{tsa}}(\mathcal{A}), \mathbf{Aut}_{\mathbf{ptls}}(\mathcal{A}_s^*))$, for any $\mathcal{A} \in \mathbf{tsa}$. $\mathbf{Q}(\mathcal{A}, \mathcal{B})$ is closed in $\mathcal{L}_s(\mathcal{A}, \mathcal{B})$ since it is so $P(\mathcal{A}, \mathcal{B})$ and $Ef(\mathcal{B})$ is closed in \mathcal{B} . $\Theta_{\mathcal{A}}$ is a subsemigroup of \mathcal{A} and if the product in \mathcal{A} is jointly continuous then $\Theta_{\mathcal{A}}$ is closed. $\delta_{\mathcal{A}}^+ : \mathcal{A} \rightarrow P(\mathcal{A}_s^*)$ and it is a semigroup morphism since (12) and Lemma 1.3.1(2) & (3). Next we introduce our definition of propensity map

DEFINITION 1.3.4. Let $\mathcal{A}, \mathcal{B} \in \mathbf{tsa}$ define

$$\mathfrak{Z}(\mathcal{A}, \mathcal{B}) := \{J \in P(\mathcal{A}_s^*, \mathcal{B}_s^*) \mid (\forall \phi \in \mathfrak{P}_{\mathcal{A}})(J(\phi)(1) \leq \phi(1))\},$$

and $\mathfrak{Z}(\mathcal{A}) := \mathfrak{Z}(\mathcal{A}, \mathcal{A})$. Define

$$\mathfrak{b}_{\mathcal{A}, \mathcal{B}} : \mathfrak{Z}(\mathcal{A}, \mathcal{B}) \times \mathfrak{P}_{\mathcal{A}}^{\mathfrak{h}} \times \text{Ef}(\mathcal{B}) \rightarrow [0, 1] \quad (J, \omega, e) \mapsto \frac{J(\omega)(e)}{\omega(1)}.$$

We shall return later in commenting this definition, for the moment we note that the propensity map $\mathfrak{b}_{\mathcal{A}, \mathcal{B}}$ is defined on triplets (J, ω, e) . Here ω will be interpreted as a statistical ensemble with strength $\omega(1)$, J is a generalized channel mapping statistical ensembles into statistical ensembles ensuring that the strength of the transformed ensembles will not increase, e is an effect in the usual interpretation. For our physical interpretation given in Def. 1.3.8, we require the algebraic information stated in the following two propositions

PROPOSITION 1.3.5. *Let \circ be the map composition then any of the following set of data determines uniquely a category*

- (1) $\langle \text{Obj}(\mathbf{tsa}), \{\mathfrak{Z}(\mathcal{A}, \mathcal{B}) \mid \mathcal{A}, \mathcal{B} \in \mathbf{tsa}\}, \circ \rangle$;
- (2) $\langle \text{Obj}(\mathbf{tsa}), \{\mathbf{Q}(\mathcal{A}, \mathcal{B}) \mid \mathcal{A}, \mathcal{B} \in \mathbf{tsa}\}, \circ \rangle$;
- (3) $\langle \{\text{Ef}(\mathcal{A}) \mid \mathcal{A} \in \mathbf{tsa}\}, \{\mathfrak{Q}(\mathcal{A}, \mathcal{B}) \mid \mathcal{A}, \mathcal{B} \in \mathbf{tsa}\}, \circ \rangle$.

PROOF. St. (1) amounts to prove that $K \circ J \in \mathfrak{Z}(\mathcal{A}, \mathcal{C})$ for any $J \in \mathfrak{Z}(\mathcal{A}, \mathcal{B}), K \in \mathfrak{Z}(\mathcal{B}, \mathcal{C})$, which follows since \mathbf{pts} is a category since for all $\phi \in \mathfrak{P}_{\mathcal{A}}$ we have $K(J(\phi))(1) \leq J(\phi)(1) \leq 1$. St. (2) follows since (7). St. (3) amounts to prove that $T(\text{Ef}(\mathcal{A})) \subseteq \text{Ef}(\mathcal{B})$ and $S \circ T \in \mathfrak{Q}(\mathcal{A}, \mathcal{C})$ for any $T \in \mathfrak{Q}(\mathcal{A}, \mathcal{B}), S \in \mathfrak{Q}(\mathcal{B}, \mathcal{C})$. The inclusion follows since T is an order morphism, being positive and linear, while the last sentence follows since (2). \square

DEFINITION 1.3.6. *Let \mathbf{pts} be the category constructed in Prp. 1.3.5(2), and let $\mathfrak{p} \in \mathbf{Fct}(\mathbf{tsa}, \mathbf{pts})$ the forgetful functor.*

PROPOSITION 1.3.7. *Let $\mathcal{A}, \mathcal{B} \in \mathbf{tsa}$, then*

- (1) $\mathbf{K}(\mathcal{A}) \subseteq \mathbf{Q}(\mathcal{A})$;
- (2) $\gamma_{\mathcal{A}, e} : \Theta_{\mathcal{A}}^{\text{op}} \rightarrow \mathfrak{Q}(\mathcal{A})$ semigroup morphism;
- (3) $\mathbf{Q}(\mathcal{B}, \mathcal{A})^{\dagger} \subseteq \mathfrak{Z}(\mathcal{A}, \mathcal{B})$;
- (4) $\gamma_{\mathcal{A}}^{\dagger} : \Theta_{\mathcal{A}} \rightarrow \mathfrak{Z}(\mathcal{A})$ semigroup morphism.

PROOF. St.(2) is well-set, $\mathfrak{Q}(\mathcal{A})$ being a semigroup since Prp. 1.3.5(3), and it follows since $\gamma_{\mathcal{A}}(\Theta_{\mathcal{A}}) \subseteq \mathbf{Q}_{\mathcal{A}}$ and (12). St.(3) follows since Lemma 1.3.1(3) and since any element in $\mathfrak{P}_{\mathcal{A}}$ is an order morphism being linear. St.(4) is well-set, $\mathfrak{Z}(\mathcal{A})$ being a semigroup since Prp. 1.3.5(1), and it follows since st.(2)&(3) and Lemma 1.3.1(2). \square

Next we introduce the physical interpretation of the previous data. In the following definition 1 is the unit of the unital algebra \mathcal{A} , while $1_{\mathcal{A}}$ is the identity map on \mathcal{A} in agreement with section 1.2.

DEFINITION 1.3.8. *We call $(\mathfrak{s}, \mathfrak{u})$ a semantics if it satisfies the following properties: for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{tsa}$, and any Hilbert space \mathfrak{H} , we have for all $J \in \mathfrak{Z}(\mathcal{A}, \mathcal{B}), K \in \mathfrak{Z}(\mathcal{B}, \mathcal{C}), T \in \mathbf{Q}(\mathcal{A}, \mathcal{B}), S \in \mathbf{Q}(\mathcal{B}, \mathcal{C}), A \in \mathcal{A}_{\text{ob}}, O \in \mathcal{L}(\mathfrak{H})_{\text{ob}}, M$ Borelian set of $\mathbb{R}, e \in \text{Ef}(\mathcal{A}), \psi \in \mathfrak{P}_{\mathcal{A}}, c, d \in \Theta_{\mathcal{A}}$, and $a, b \in \Theta_{\mathcal{A}}$ such that $a + b \in \Theta_{\mathcal{A}}$*

- (1) $\mathfrak{s}(dc) \equiv \mathfrak{s}(d)$ after $\mathfrak{s}(c)$,
- (2) $\mathfrak{s}(a + b) \equiv \mathfrak{s}(a)$ in parallel with $\mathfrak{s}(b)$,
- (3) $\mathfrak{s}(S \circ T) \equiv \mathfrak{s}(S)$ following $\mathfrak{s}(T)$,
- (4) $\mathfrak{s}(S \circ T) \equiv \mathfrak{s}(T)$ followed by $\mathfrak{s}(S)$,
- (5) $\mathfrak{s}(K \circ J) \equiv \mathfrak{s}(K)$ following $\mathfrak{s}(J)$,
- (6) $\mathfrak{s}(K \circ J) \equiv \mathfrak{s}(J)$ followed by $\mathfrak{s}(K)$,
- (7) $\mathfrak{u}(0) =$ producing no outputs,
- (8) $\mathfrak{u}(1) =$ producing one output,
- (9) $\mathfrak{s}(e) \equiv$ the effect $\mathfrak{u}(e)$,
- (10) $\mathfrak{u}(E_O(M)) =$ of obtaining a value in M when measuring $\mathfrak{s}(O)$,
- (11) $\mathfrak{u}(c^*c) =$ produced by $\mathfrak{s}(c)$,
- (12) $\mathfrak{u}(1_A) =$ producing no variations,
- (13) $\mathfrak{s}(1) \equiv$ the strength⁸,
- (14) $\mathfrak{s}(c) =$ the operation $\mathfrak{u}(c)$,
- (15) $\mathfrak{u}(e^{1/2}) =$ producing $\mathfrak{s}(e)$,
- (16) $\mathfrak{s}(T) =$ the device $\mathfrak{u}(T)$,
- (17) $\mathfrak{u}(\gamma_A(c)) =$ implementing $\mathfrak{s}(c)$,
- (18) $\mathfrak{s}(A) =$ the observable $\mathfrak{u}(A)$
- (19) $\mathfrak{u}(T(A)) \equiv$ resulting next $\mathfrak{s}(T)$ interacts with $\mathfrak{s}(A)$,
- (20) $\mathfrak{u}(T(A)) \equiv$ resulting next $\mathfrak{s}(A)$ interacts with $\mathfrak{s}(T)$,
- (21) $\mathfrak{s}(J) =$ the channel $\mathfrak{u}(J)$,
- (22) $\mathfrak{u}(T^\dagger) \equiv$ induced by $\mathfrak{s}(T)$,
- (23) $\mathfrak{s}(\psi) =$ the statistical ensemble $\mathfrak{u}(\psi)$,
- (24) $\mathfrak{u}(J(\psi)) \equiv$ resulting next $\mathfrak{s}(J)$ interacts with any sample of $\mathfrak{s}(\psi)$,
- (25) $\mathfrak{u}(J(\psi)) \equiv$ resulting next any sample of $\mathfrak{s}(\psi)$ interacts with $\mathfrak{s}(J)$;

where E_O is the resolution of the identity of O . Let $(\mathfrak{s}, \mathfrak{u})$ be a semantics and $\mathcal{A}, \mathcal{B} \in \mathbf{tsa}$, thus for any $\psi \in \mathfrak{P}_{\mathcal{A}}$, $A \in \mathcal{A}_{ob}$ and any $J \in \mathfrak{Z}(\mathcal{A}, \mathcal{B})$, $\omega \in \mathfrak{P}_{\mathcal{A}}^{\mathfrak{b}}$ and $e \in Ef(\mathcal{B})$ we say that

- (1) $\psi(A)$ equals the measure in $\mathfrak{s}(\psi)$ of $\mathfrak{s}(A)$;
- (2) $\mathfrak{b}_{\mathcal{A}, \mathcal{B}}(J, \omega, e)$ equals **the empirical representation of the propensity conditioned by $\mathfrak{s}(\omega)$ to obtain $\mathfrak{s}(e)$ when tested on any sample of $\mathfrak{s}(J(\omega))$.**

In the remaining of the paper, unless the contrary is not stated and except Rmk. 1.3.9, we let the propensity mean the empirical representation of the propensity. Non-normalized statistical ensembles do not pose in our framework any interpretational issue, since multiplying a statistical ensemble ω by a positive scalar b results in rescaling the measure of the strength in ω by the factor b Rmk. 1.3.12. Operatively the measure of the strength in a statistical ensemble is a primitive concept upon which the remaining ones can be established. Indeed always propensities and in the most physically important cases also observables, all the cases if the observable algebra is a C^* -algebra, can be stated in terms of measure of strength in suitable statistical ensembles see Rmk. 1.3.12.

⁸with the same meaning of strength of a beam discussed in [11].

REMARK 1.3.9 (Propensity versus probability and the role of time). As we anticipated we retain the concept of propensity primitive and predetermined with respect to any measure, while we consider the value $b_{\mathcal{A},\mathcal{B}}(J, \omega, e)$ as its experimentally testable representative. As a result all the empirical characters of $b_{\mathcal{A},\mathcal{B}}(J, \omega, e)$ are ascribable to the representative rather than to the propensity itself. We do this because the concept of frequency is strongly related to and dependent by the concept of time. Indeed by the very definition of frequency performing trials implies at least an operative meaning of time labelling them, and this results to supply the theory with a primitive concept of time. However, as it will be clear in the course of the paper, here we do not assume any global nor primitive concept of time. Rather as advanced in the introduction, in our framework the couple formed by a species \mathbf{a} of dynamical patterns and a context M in the context category source of \mathbf{a} , determines a collection of experimentally detectable trajectories whose dynamics is implemented by the morphism part $\tau_{\mathbf{a}(M)}$ of the dynamical functor acting over the morphisms of the corresponding dynamical category $G_M^{\mathbf{a}}$. Thus we can read $Mor_{G_M^{\mathbf{a}}}$ as a sort of proper time associated with the species \mathbf{a} in the context M .

REMARK 1.3.10 (The propensity map is well-defined). Let $\mathcal{A}, \mathcal{B} \in \mathbf{tsa}$, $J \in \mathcal{Z}(\mathcal{A}, \mathcal{B})$, $\omega \in \mathfrak{P}_{\mathcal{A}}$ and $e \in Ef(\mathcal{B})$. Thus $J(\omega) \in \mathfrak{P}_{\mathcal{B}}$ such that $J(\omega)(e) \in [0, \omega(1)]$, indeed $J(\omega)(e) \in [0, J(\omega)(1)]$ since $J(\omega) \in \mathfrak{P}_{\mathcal{B}}$, the remaining follows by the definition of $\mathcal{Z}(\mathcal{A}, \mathcal{B})$. Therefore $b_{\mathcal{A},\mathcal{B}}$ is well-defined as a map into $[0, 1]$ so compatible with the probabilistic interpretation given in Def. 1.3.8. According the meaning given in Def. 1.3.8, J is a transformation of statistical ensembles not increasing the strength however it is not excluded that J filters all the samples of an ensemble ω , i.e. $J(\omega)(1) = 0$. Here statistical ensemble means an ensemble of preparations (samples) realized by macroscopic apparatuses under well defined repeatable conditions, which is basically the standard meaning see for instance in [10, p.116].

REMARK 1.3.11 (The propensity map slightly generalizes the state-effect duality). Let $p_{\mathcal{A}} : \mathfrak{E}_{\mathcal{A}} \times Ef(\mathcal{A}) \rightarrow [0, 1]$ be $(\psi, u) \mapsto \psi(u)$, so $p_{\mathcal{A}}$ is the usual state-effect duality if \mathcal{A} is a C^* -algebra. Next let $J \in \mathcal{Z}(\mathcal{A}, \mathcal{B})$ and $\omega \in \mathfrak{P}_{\mathcal{A}}^{\natural}$ thus for all $e \in Ef(\mathcal{B})$ we have

$$b_{\mathcal{A},\mathcal{B}}(J, \omega, e) \leq b_{\mathcal{B},\mathcal{B}}(1_{\mathcal{B}^*}, J(\omega), e) = p_{\mathcal{B}}\left(\frac{J(\omega)}{J(\omega)(1)}, e\right),$$

in particular $b_{\mathcal{B},\mathcal{B}}$ implements $p_{\mathcal{B}}$ and $b_{\mathcal{A},\mathcal{B}}(J, \omega, e)$ has no counterpart in terms of $p_{\mathcal{B}}$ unless $J \in \mathcal{Q}(\mathcal{B}, \mathcal{A})^{\dagger}$ or $J(\omega)(1) = \omega(1)$ see below; hence $b_{\mathcal{A},\mathcal{B}}$ generalizes $p_{\mathcal{B}}$. The reason is clear since in the left side of the inequality the propensity is conditioned by the stronger statistical ensemble ω , while in both sides of the equality the propensity is conditioned by the weaker statistical ensemble $J(\omega)$. Moreover it is easy to see that (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4) \Leftrightarrow (5), where

- (1) $J \in \mathcal{Q}(\mathcal{B}, \mathcal{A})^{\dagger}$;
- (2) $(\exists T \in \mathcal{Q}(\mathcal{B}, \mathcal{A}))(\forall \phi \in \mathfrak{P}_{\mathcal{A}}^{\natural})(\forall e \in Ef(\mathcal{B}))(b_{\mathcal{A},\mathcal{B}}(J, \phi, e) = p_{\mathcal{A}}(\frac{\phi}{\phi(1)}, T(e)))$;
- (3) $J(\omega)(1) = \omega(1)$;
- (4) $(\forall e \in Ef(\mathcal{B}))(b_{\mathcal{A},\mathcal{B}}(J, \omega, e) = p_{\mathcal{B}}(\frac{J(\omega)}{J(\omega)(1)}, e))$;

$$(5) (\exists e \in Ef(\mathcal{B}))(\mathfrak{b}_{\mathcal{A},\mathcal{B}}(J, \omega, e) = p_{\mathcal{B}}(\frac{J(\omega)}{J(\omega)(1)}, e));$$

where (2) is well-set since Prp. 1.3.7(3) and Prp. 1.3.5(3). Note that the restriction $T(1) = 1$ allows to read $p_{\mathcal{A}}(\psi, T(e))$ in terms of state transformation $p_{\mathcal{A}}(T^{\dagger}(\psi), e)$. However this restriction leaves outside for instance all the channels conjugate of operations, channels used for example to generate the interference phenomenon, as we shall see Rmk. 1.3.14. Here the term channel is used in the broader sense of function mapping statistical ensembles say on \mathcal{A} into statistical ensembles say in \mathcal{B} , namely an element of $\mathfrak{Z}(\mathcal{A}, \mathcal{B})$.

REMARK 1.3.12 (Propensity and measure of observables in terms of measure of strength. Mean value in terms of propensities). Let us adapt to our framework the standard definition of mean value, namely let $\mathcal{A} \in \mathfrak{tsa}$, $\omega \in \mathfrak{P}_{\mathcal{A}}^{\natural}$ and $O \in \mathcal{A}_{ob}$, define the mean value of $\mathfrak{s}(O)$ in $\mathfrak{s}(\omega)$ to be the real number $mv(\omega, O) := \omega(O)/\omega(1)$. Next let $\mathcal{A}, \mathcal{B} \in \mathfrak{tsa}$ and $\omega \in \mathfrak{P}_{\mathcal{A}}$, then $\omega(1)$ equals the measure of the strength in the statistical ensemble $\mathfrak{u}(\omega)$, in particular any positive scalar multiple $b\omega$ represents a statistical ensemble obtained from ω by rescaling its strength of the factor b . Next let $e \in Ef(\mathcal{B})$ and $J \in \mathfrak{Z}(\mathcal{A}, \mathcal{B})$ then $J(\omega)(e) = (\gamma_{e^{1/2}}^{\dagger} \circ J)(\omega)(1)$. Hence $J(\omega)(e)$ equals the measure of the strength in the statistical ensemble resulting next, the channel $\mathfrak{u}(J)$ followed by the channel induced by the device implementing the operation producing the effect $\mathfrak{u}(e)$, interacts with any sample of the statistical ensemble $\mathfrak{u}(\omega)$. If $\omega(1) \neq 0$, then the propensity is just a ratio of two measures of strength indeed

$$\mathfrak{b}_{\mathcal{A},\mathcal{B}}(J, \omega, e) = \omega(1)^{-1}(\gamma_{e^{1/2}}^{\dagger} \circ J)(\omega)(1).$$

Next let \mathcal{A} be a C^* -algebra, $O \in \mathcal{A}_{ob}$, $\omega \in \mathfrak{P}_{\mathcal{A}}^{\natural}$ and $\langle \mathfrak{K}, \eta, \xi \rangle$ be the GNS construction associated with ω . Let T_{ω}^O be the resolution of the identity of the bounded selfadjoint operator $\eta(O)$ and ω_{ξ} the vector functional on $\mathcal{L}(\mathfrak{K})$ induced by the vector ξ . Thus there exists a measure \mathfrak{v}_{ω}^O on \mathbb{R} whose support is the spectrum of $\eta(O)$ and such that

$$\begin{aligned} \omega(O) &= \int \lambda d\mathfrak{v}_{\omega}^O(\lambda) \\ \mathfrak{v}_{\omega}^O(\sigma) &= \gamma_{T_{\omega}^O(\sigma)}^{\dagger}(\omega_{\xi})(1_{\mathfrak{K}}), \forall \sigma \in \mathfrak{B}(\mathbb{R}); \end{aligned}$$

where $\mathfrak{B}(\mathbb{R})$ is the collection of the Borelian subsets of \mathbb{R} . Therefore the measure $\omega(O)$ of the observable $\mathfrak{u}(O)$ in the statistical ensemble $\mathfrak{u}(\omega)$ reads in terms of the measure of the strength in the statistical ensembles $\mathfrak{u}(\gamma_{T_{\omega}^O(\sigma)}^{\dagger}(\omega_{\xi}))$. In particular $\omega(O)$ is operatively well-defined by these measures of strength.

Similarly let $\langle \mathfrak{H}, \pi, \Omega \rangle$ be the GNS construction associated with the state $\psi := \omega(1)^{-1}\omega$. Let E_{ω}^O be the resolution of the identity of the bounded selfadjoint operator $\pi(O)$ and ω_{Ω} the vector state on $\mathcal{L}(\mathfrak{H})$ induced by the unit vector Ω . Thus there exists a probability measure μ_{ω}^O on \mathbb{R} whose support is the spectrum of $\pi(O)$ and such that

$$\begin{aligned} \omega(O) &= \omega(1) \int \lambda d\mu_{\omega}^O(\lambda) \\ \mu_{\omega}^O(\sigma) &= \gamma_{E_{\omega}^O(\sigma)}^{\dagger}(\omega_{\Omega})(1_{\mathfrak{H}}), \forall \sigma \in \mathfrak{B}(\mathbb{R}). \end{aligned}$$

Therefore the measure $\omega(O)$ of the observable $u(O)$ in the statistical ensemble $u(\omega)$ reads in terms of the measure of the strength in the statistical ensemble $u(\omega)$ and in terms of the measure of the strength in the statistical ensembles $u(\gamma_{E_\omega^O(\sigma)}^+(\omega_\Omega))$. In particular $\omega(O)$ is operatively well-defined by these measures of strength. Notice that $u(\gamma_{E_\omega^O(\sigma)}^+(\omega_\Omega)) \equiv$ resulting next any sample of the statistical ensemble $u(\omega_\Omega)$ interacts with the channel induced by the device implementing the operation producing the effect of obtaining a value in σ when measuring the observable $u(\pi(O))$. We deduce from the above equality that the mean value, consistently with the usual meaning ascribed to it, reads as follows

$$mv(\omega, O) = \int \lambda d\mu_\omega^O(\lambda),$$

$$\mu_\omega^O(\sigma) = p_{\mathcal{L}(\mathfrak{S})}(\omega_\Omega, E_\omega^O(\sigma)), \forall \sigma \in \mathfrak{B}(\mathbb{R});$$

where $p_{\mathcal{L}(\mathfrak{S})}(\omega_\Omega, E_\omega^O(\sigma))$ is the propensity conditioned by the statistical ensemble $u(\omega_\Omega)$ to obtain the effect of obtaining a value in σ when measuring the observable $u(\pi(O))$. Finally assume that \mathcal{A} is an additive group, 1 is a preferred element and that for any suitable additive functional ω on \mathcal{A} the operational meaning of the mean value $mv(\omega, O)$ defined as above, is well-defined and that $\omega(1)$ maintains the meaning of the measure of the strength. Then via the formula $\omega(O) = mv(\omega, O)\omega(1)$ we deduce that the operational meaning of $\omega(O)$ is well-defined.

REMARK 1.3.13. Under the notation of Def. 1.3.8 we have that $s(\gamma_{\mathcal{A}}^+(c)) \equiv$ the channel induced by the device implementing $s(c)$, hence $s(\gamma_{\mathcal{A}}^+(c(a+b))) \equiv$ the channel induced by the device implementing the operation $u(c)$ after, the operation a in parallel with the operation $u(b)$, and $s(\gamma_{\mathcal{A}}^+(ca)) \equiv$ the channel induced by the device implementing the operation $u(c)$ after the operation $u(a)$.

REMARK 1.3.14 (Interference phenomenon). We apply the semantics to the well-known phenomenon of interference. Let $\mathcal{A} \in \mathbf{tsa}$, and for all $a, b, c \in \mathcal{A}$ set $Int(a, b, c) := a^*c^*cb + b^*c^*ca$, so if $\{a, b\} \subset \mathbb{C}\{c^*c\}'$ then in general $Int(a, b, c) \neq 0$ even in case $a^*b = 0$. Next for any $a_1, a_2 \in \Theta_{\mathcal{A}}$ such that $a_1 + a_2 \in \Theta_{\mathcal{A}}$, we have

$$b(\gamma_{\mathcal{A}}^+(c(a_1 + a_2)), \omega, 1) = \sum_{i=1}^2 b(\gamma_{\mathcal{A}}^+(ca_i), \omega, 1) + \frac{\omega(Int(a, b, c))}{\omega(1)}.$$

Therefore according Rmk. 1.3.13 and Def. 1.3.8 we can state that the propensity conditioned by the statistical ensemble $u(\omega)$, to obtain an output when tested on any sample of the statistical ensemble resulting next the channel induced by the device implementing the operation c after, the operation a in parallel with the operation b , interacts with any sample of the statistical ensemble $u(\omega)$, differs of the ammount $\omega(Int(a, b, c))/\omega(1)$ from the sum $p_1 + p_2$, where p_i is the propensity conditioned by the statistical ensemble $u(\omega)$, to obtain an output when tested on any sample of the statistical ensemble resulting next the channel induced by the device implementing the operation c after the operation a_i , interacts with any sample of the statistical ensemble $u(\omega)$, for any $i \in \{1, 2\}$.

1.4. Species of dynamical patterns and equiformity principle

We introduce in **Def. 1.4.1** the concept of dynamical pattern and its transformations, the building block of all the constructions of this paper. In **Rmk. 1.4.3** we make explicit the definition and show in **Cor. 1.4.5** that dynamical patterns form a category \mathfrak{dp} . In **Def. 1.4.7**, **Rmk. 1.4.9** and **Cor. 1.4.11** we introduce, explain and organize in a category the concept of preordered dynamical pattern, employed to address the dual of a dynamical pattern via the construction in **Cor. 1.4.16** of a controvariant functor. In **Thm. 1.4.20** we prove the existence of a functor from \mathfrak{dp} to the category \mathfrak{Chdv} of channels and devices introduced in **Def. 1.4.18** and **Cor. 1.4.19**. \mathfrak{Chdv} is essential in order to extract empirical information from \mathfrak{dp} . We consider a species of dynamical patterns contextualized in a category \mathfrak{D} to be a functor from \mathfrak{D} to \mathfrak{Chdv} . In **Def. 1.4.22** and **Def. 1.4.23** we define collections of trajectories associated with any species which encode the dynamical information of the species. Experimental settings are introduced in **Def. 1.4.24**, while in **Def. 1.4.27** we define a link between experimental settings, an auxiliary concept in order to express the fundamental equiformity principle in **Prp. 1.4.34**. In **Def. 1.4.29** we present the physical interpretation of the data. **Lemma 1.4.41** prepares to the main result of this section and one of the entire paper namely **Thm. 1.4.46**. There we prove that given any connector T , a natural transformation between species, then for any experimental setting \mathfrak{Q} of its target species and any function s associated with \mathfrak{Q} and T , there exists an experimental setting $T[\mathfrak{Q}, s]$ of its source species, such that T is a link from \mathfrak{Q} to $T[\mathfrak{Q}, s]$, so inducing an equiformity principle. With the price of coarsening the equiformity principle to the standard experimental setting of the source species we can eliminate the degeneration in **s Thm. 1.4.47**. Finally we establish in the second and third main result that vertical composition of connectors is controvariantly represented as charge composition **Cor. 1.4.48** and that horizontal composition is represented as charge transfer **Cor. 1.4.53**. We recall that \mathfrak{tsa} and \mathfrak{ptls} are top-quasi enriched categories.

DEFINITION 1.4.1. \mathfrak{A} is called dynamical pattern, shortly \mathfrak{dp} , if $\mathfrak{A} = \langle W, \eta \rangle$ where W is a top-quasi enriched category and $\eta \in \mathbf{Fct}_{\text{top}}(W, \mathfrak{tsa})$. Let $\mathfrak{A} = \langle W, \eta \rangle$ be a \mathfrak{dp} , then we denote W by $G_{\mathfrak{A}}$, $\text{Mor}_W(x, y)$ by $G_{\mathfrak{A}}(x, y)$, for all $x, y \in W$, η by $\sigma_{\mathfrak{A}}$, while the object and morphism maps η_o and η_m by $A_{\mathfrak{A}}$ and $\tau_{\mathfrak{A}}$ respectively. Let $\mathfrak{A}, \mathfrak{B}$ and \mathfrak{C} be \mathfrak{dp} , define $1_{\mathfrak{A}} := (1_{G_{\mathfrak{A}}}, 1_{\sigma_{\mathfrak{A}}})$ and

$$(13) \quad \text{Mor}_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B}) := \coprod_{f \in \mathbf{Fct}_{\text{top}}(G_{\mathfrak{B}}, G_{\mathfrak{A}})} \text{Mor}_{\mathbf{Fct}(G_{\mathfrak{B}}, \mathfrak{tsa})}(\sigma_{\mathfrak{A}} \circ f, \sigma_{\mathfrak{B}}),$$

and

$$(14) \quad \begin{aligned} (\circ) : \text{Mor}_{\mathfrak{dp}}(\mathfrak{B}, \mathfrak{C}) \times \text{Mor}_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B}) &\rightarrow \text{Mor}_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{C}), \\ (g, S) \circ (f, T) &:= (f \circ g, S \circ (T * 1_g)). \end{aligned}$$

\mathfrak{A} is an \mathcal{U} -type dynamical pattern or \mathcal{U} -type \mathfrak{dp} if it is a \mathfrak{dp} such that $\text{Obj}(G_{\mathfrak{A}}) \simeq A$ and $A \subseteq \mathcal{U}$.

Often we call $G_{\mathfrak{A}}$ the dynamical category of \mathfrak{A} and $\sigma_{\mathfrak{A}}$, as well by abuse of language $\tau_{\mathfrak{A}}$, the dynamical functor of \mathfrak{A} . This definition nontrivially extends the category of dynamical systems as we shall see in [31], where we specialize to the subcategory of those dynamical patterns whose dynamical category is the groupoid associated with

a topological group, and thus the dynamical functor reduces to a representation of the topological group in terms of $*$ -automorphisms of a $*$ -topological algebra, while morphisms are couples formed by a continuous group morphism and an equivariant map between the respective representations.

REMARK 1.4.2. If \mathfrak{A} is a \mathfrak{dp} then according the notations in Def. 1.4.1 we have $\mathfrak{A} = \langle G_{\mathfrak{A}}, \sigma_{\mathfrak{A}} \rangle = \langle (Obj(G_{\mathfrak{A}}), \{G_{\mathfrak{A}}(x, y)\}_{x, y \in Obj(G_{\mathfrak{A}})}), (\mathcal{A}_{\mathfrak{A}}, \tau_{\mathfrak{A}}) \rangle$. If in addition \mathfrak{A} is of \mathcal{U} -type then $G_{\mathfrak{A}}$ is an \mathcal{U} -type category in particular is an object of \mathbf{cat} .

Next we provide a decodification of Def. 1.4.1.

REMARK 1.4.3. Let $Mor_{\mathbf{tsa}}(X, Y)$ be provided with the topology of pointwise convergence for all $X, Y \in \mathbf{tsa}$. Thus $\langle (Obj(G_{\mathfrak{A}}), \{G_{\mathfrak{A}}(x, y)\}_{x, y \in Obj(G_{\mathfrak{A}})}), (\mathcal{A}_{\mathfrak{A}}, \tau_{\mathfrak{A}}) \rangle$ is a \mathfrak{dp} iff

- (1) $G_{\mathfrak{A}}$ is a \mathcal{U} -category such that $G_{\mathfrak{A}}(x, y)$ is a topological space and the morphism composition $\circ : G_{\mathfrak{A}}(y, z) \times G_{\mathfrak{A}}(x, y) \rightarrow G_{\mathfrak{A}}(x, z)$ is a separatedly continuous map, for all $x, y, z \in Obj(G_{\mathfrak{A}})$;
- (2) $\mathcal{A}_{\mathfrak{A}} : Obj(G_{\mathfrak{A}}) \rightarrow Obj(\mathbf{tsa})$;
- (3) $\tau_{\mathfrak{A}} : Mor_{G_{\mathfrak{A}}} \rightarrow Mor_{\mathbf{tsa}}$ such that $\tau_{\mathfrak{A}}^{y, z} : G_{\mathfrak{A}}(y, z) \rightarrow Mor_{\mathbf{tsa}}(\mathcal{A}_{\mathfrak{A}}(y), \mathcal{A}_{\mathfrak{A}}(z))$ is a continuous map, for all $y, z \in Obj(G_{\mathfrak{A}})$;
- (4) $\tau_{\mathfrak{A}}(g \circ h) = \tau_{\mathfrak{A}}(g) \circ \tau_{\mathfrak{A}}(h)$, and $\tau_{\mathfrak{A}}(1_x) = 1_{\mathcal{A}_{\mathfrak{A}}(x)}$, for all $x, y, z \in Obj(G_{\mathfrak{A}})$, $g \in G_{\mathfrak{A}}(y, z)$ and $h \in G_{\mathfrak{A}}(x, y)$.

Let \mathfrak{A} and \mathfrak{B} be \mathfrak{dp} , thus $(f, T) \in Mor_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B})$ iff

- (1) $f = (f_o, f_m)$ such that $f_o : Obj(G_{\mathfrak{B}}) \rightarrow Obj(G_{\mathfrak{A}})$ and $f_m : Mor_{G_{\mathfrak{B}}} \rightarrow Mor_{G_{\mathfrak{A}}}$;
- (2) for all $y, z \in Obj(G_{\mathfrak{B}})$
 - (a) $f_m^{y, z} : G_{\mathfrak{B}}(y, z) \rightarrow G_{\mathfrak{A}}(f_o(y), f_o(z))$ is a continuous map;
 - (b) $f_m(g \circ h) = f_m(g) \circ f_m(h)$, for all $x \in Obj(G_{\mathfrak{B}})$, $g \in G_{\mathfrak{B}}(y, z)$ and $h \in G_{\mathfrak{B}}(x, y)$;
 - (c) $T \in \prod_{x \in Obj(G_{\mathfrak{B}})} Mor_{\mathbf{tsa}}(\mathcal{A}_{\mathfrak{A}}(f_o(x)), \mathcal{A}_{\mathfrak{B}}(x))$ such that for all $g \in G_{\mathfrak{B}}(y, z)$ we have that the following diagram in \mathbf{tsa} is commutative

$$\begin{array}{ccc}
 \mathcal{A}_{\mathfrak{A}}(f_o(y)) & \xrightarrow{T(y)} & \mathcal{A}_{\mathfrak{B}}(y) \\
 \tau_{\mathfrak{A}}(f_m(g)) \downarrow & & \downarrow \tau_{\mathfrak{B}}(g) \\
 \mathcal{A}_{\mathfrak{A}}(f_o(z)) & \xrightarrow{T(z)} & \mathcal{A}_{\mathfrak{B}}(z)
 \end{array}$$

In order to prove in Cor. 1.4.5 that dynamical patterns and their transformations form a category we need the following

LEMMA 1.4.4. *The composition in (14) is a well-defined associative map such that $(f, T) \circ 1_{\mathfrak{A}} = 1_{\mathfrak{B}} \circ (f, T) = (f, T)$, for all \mathfrak{A} and \mathfrak{B} \mathfrak{dp} and $(f, T) \in Mor_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B})$.*

PROOF. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and \mathfrak{D} be \mathfrak{dp} , $(f, T) \in Mor_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B})$, $(g, S) \in Mor_{\mathfrak{dp}}(\mathfrak{B}, \mathfrak{C})$, and $(h, V) \in Mor_{\mathfrak{dp}}(\mathfrak{C}, \mathfrak{D})$. Then $f \in \mathbf{Fct}_{\mathbf{top}}(G_{\mathfrak{B}}, G_{\mathfrak{A}})$, $T \in Mor_{\mathbf{Fct}(G_{\mathfrak{B}}, \mathbf{tsa})}(\sigma_{\mathfrak{A}} \circ f, \sigma_{\mathfrak{B}})$, and $g \in \mathbf{Fct}_{\mathbf{top}}(G_{\mathfrak{C}}, G_{\mathfrak{B}})$,

$S \in \text{Mor}_{\text{Fct}(G_{\mathbb{C}}, \text{tsa})}(\sigma_{\mathbb{B}} \circ g, \sigma_{\mathbb{C}})$. Thus $f \circ g \in \text{Fct}_{\text{top}}(G_{\mathbb{C}}, G_{\mathbb{A}})$, and $T * 1_g \in \text{Mor}_{\text{Fct}(G_{\mathbb{C}}, \text{tsa})}(\sigma_{\mathbb{A}} \circ f \circ g, \sigma_{\mathbb{B}} \circ g)$ hence $S \circ (T * 1_g) \in \text{Mor}_{\text{Fct}(G_{\mathbb{C}}, \text{tsa})}(\sigma_{\mathbb{A}} \circ f \circ g, \sigma_{\mathbb{C}})$ which prove that the composition in (14) is well-defined. Next $((h, V) \circ (g, S)) \circ (f, T) = (fgh, V \circ (S * 1_h) \circ (T * 1_{gh}))$, while $(h, V) \circ ((g, S) \circ (f, T)) = (fgh, V \circ ((S \circ (T * 1_g)) * 1_h))$. Next we have

$$\begin{aligned} (S * 1_h) \circ (T * 1_{gh}) &= (S * 1_h) \circ ((T * 1_g) * 1_h) \\ &= (S \circ (T * 1_g)) * (1_h \circ 1_h) = (S \circ (T * 1_g)) * 1_h, \end{aligned}$$

where in the first equality we used $1_{gh} = 1_g * 1_h$ and the associativity of $*$, the second equality follows since (1); thus the composition in (14) is associative. Next $(f, T) \circ 1_{\mathbb{A}} = (1_{G_{\mathbb{A}}} \circ f, T \circ (1_{\sigma_{\mathbb{A}}} * 1_f))$, and $1_{\mathbb{B}} \circ (f, T) = (f \circ 1_{G_{\mathbb{B}}}, 1_{\sigma_{\mathbb{B}}} \circ (T * 1_{1_{G_{\mathbb{B}}}}))$. Moreover $1_{\sigma_{\mathbb{A}}} * 1_f = 1_{\sigma_{\mathbb{A}} \circ f}$ so $T \circ (1_{\sigma_{\mathbb{A}}} * 1_f) = T$, and $1_{\sigma_{\mathbb{B}}} \circ (T * 1_{1_{G_{\mathbb{B}}}}) = T * 1_{1_{G_{\mathbb{B}}}} = T \circ (1_{G_{\mathbb{B}}})_o = T$ since (2), therefore $(f, T) \circ 1_{\mathbb{A}} = 1_{\mathbb{B}} \circ (f, T) = (f, T)$. \square

COROLLARY 1.4.5. *There exists a unique \mathcal{U}_0 -type category \mathfrak{dp} such that $\text{Obj}(\mathfrak{dp})$ is the set of all \mathcal{U} -type dynamical patterns, for any $\mathbb{A}, \mathbb{B} \in \text{Obj}(\mathfrak{dp})$ the set of morphisms of \mathfrak{dp} from \mathbb{A} to \mathbb{B} is the set in (13), and the law of composition of morphisms of \mathfrak{dp} is the map in (14). $1_{\mathfrak{D}}$ defined in Def. 1.4.1 is the unit morphism in \mathfrak{dp} relative to \mathfrak{D} , for all $\mathfrak{D} \in \text{Obj}(\mathfrak{dp})$. In particular \mathfrak{dp} is an object of \mathbf{Cat} .*

PROOF. The existence and uniqueness of the category \mathfrak{dp} follows since Lemma 1.4.4. Let \mathbb{A}, \mathbb{B} be objects of \mathfrak{dp} then $G_{\mathbb{A}}$ and $G_{\mathbb{B}}$ are \mathcal{U} -type categories hence \mathcal{U}_0 -small categories, similarly tsa is a \mathcal{U}_0 -small category, therefore by Prp. 1.2.1(3)

$$\begin{cases} \sigma_{\mathbb{A}}, \sigma_{\mathbb{B}} \in \mathcal{U}_0, \\ \text{Obj}(\text{Fct}_{\text{top}}(G_{\mathbb{B}}, G_{\mathbb{A}})) \in \mathcal{U}_0, \\ \text{Mor}_{\text{Fct}(G_{\mathbb{B}}, \text{tsa})}(\sigma_{\mathbb{A}} \circ f, \sigma_{\mathbb{B}}) \in \mathcal{U}_0, \forall f \in \text{Fct}_{\text{top}}(G_{\mathbb{B}}, G_{\mathbb{A}}); \end{cases}$$

which together [17, Def. 1.1.1.(vi,ix,x)] and [17, Def. 1.1.1.(iii,v)] yield $\mathbb{A}, \mathbb{B} \in \mathcal{U}_0$ and $\text{Mor}_{\mathfrak{dp}}(\mathbb{A}, \mathbb{B}) \in \mathcal{U}_0$ respectively, which proves that \mathfrak{dp} is a \mathcal{U}_0 -type category. \square

DEFINITION 1.4.6. *Let \mathfrak{dp}_{\star} be the full subcategory of \mathfrak{dp} whose object set is the subset of the $\mathbb{A} \in \text{Obj}(\mathfrak{dp})$ such that $G_{\mathbb{A}}$ is a groupoid whose inversion map is continuous.*

Note that if $\mathbb{A} \in \mathfrak{dp}_{\star}$ then $\tau_{\mathbb{A}}(g^{-1}) = \tau_{\mathbb{A}}(g)^{-1}$ for all $g \in G_{\mathbb{A}}$.

DEFINITION 1.4.7. *A is called preordered dynamical pattern, shortly \mathfrak{pdp} , if $A = \langle W, \eta \rangle$ where W is a top -quasi enriched category and $\eta \in \text{Fct}_{\text{top}}(W, \text{ptls})$. Let $A = \langle W, \eta \rangle$ be a \mathfrak{pdp} , then we denote W by G_A , $\text{Mor}_W(x, y)$ by $G_A(x, y)$, for all $x, y \in W$, η by σ_A , while the object and morphism maps η_o and η_m by X_A and τ_A respectively. Let A, B and C be \mathfrak{pdp} , define $1_A := (1_{G_A}, 1_{\sigma_A})$ and*

$$(15) \quad \text{Mor}_{\mathfrak{pdp}}(A, B) := \coprod_{f \in \text{Fct}_{\text{top}}(G_A, G_B)} \text{Mor}_{\text{Fct}(G_A, \text{ptls})}(\sigma_A, \sigma_B \circ f),$$

and

$$(16) \quad \begin{aligned} (\circ) : \text{Mor}_{\text{pdp}}(\mathbf{B}, \mathbf{C}) \times \text{Mor}_{\text{pdp}}(\mathbf{A}, \mathbf{B}) &\rightarrow \text{Mor}_{\text{pdp}}(\mathbf{A}, \mathbf{C}), \\ (g, S) \circ (f, T) &:= (g \circ f, (S * 1_f) \circ T). \end{aligned}$$

\mathbf{A} is an \mathcal{U} -type preordered dynamical pattern or \mathcal{U} -type pdp if it is a pdp such that $\text{Obj}(\mathbf{G}_{\mathbf{A}}) \simeq A$ and $A \subseteq \mathcal{U}$.

REMARK 1.4.8. If \mathbf{A} is a pdp then according the notations in the previous definition we have $\mathbf{A} = \langle \mathbf{G}_{\mathbf{A}}, \sigma_{\mathbf{A}} \rangle = \langle (\text{Obj}(\mathbf{G}_{\mathbf{A}}), \{G_{\mathbf{A}}(x, y)\}_{x, y \in \text{Obj}(\mathbf{G}_{\mathbf{A}})}), (X_{\mathbf{A}}, \tau_{\mathbf{A}}) \rangle$. If in addition \mathbf{A} is of \mathcal{U} -type then $\mathbf{G}_{\mathbf{A}}$ is an \mathcal{U} -type category in particular is an object of cat .

REMARK 1.4.9. Let $\text{Mor}_{\text{ptls}}(X, Y)$ be provided with the topology of pointwise convergence for all $X, Y \in \text{ptls}$. Thus $\langle (\text{Obj}(\mathbf{G}_{\mathbf{A}}), \{G_{\mathbf{A}}(x, y)\}_{x, y \in \text{Obj}(\mathbf{G}_{\mathbf{A}})}), (X_{\mathbf{A}}, \tau_{\mathbf{A}}) \rangle$ is a pdp iff

- (1) $\mathbf{G}_{\mathbf{A}}$ is a \mathcal{U} -category such that $G_{\mathbf{A}}(x, y)$ is a topological space and the morphism composition $\circ : G_{\mathbf{A}}(y, z) \times G_{\mathbf{A}}(x, y) \rightarrow G_{\mathbf{A}}(x, z)$ is a separatedly continuous map, for all $x, y, z \in \text{Obj}(\mathbf{G}_{\mathbf{A}})$;
- (2) $X_{\mathbf{A}} : \text{Obj}(\mathbf{G}_{\mathbf{A}}) \rightarrow \text{Obj}(\text{ptls})$;
- (3) $\tau_{\mathbf{A}} : \text{Mor}_{\mathbf{G}_{\mathbf{A}}} \rightarrow \text{Mor}_{\text{ptls}}$ such that $\tau_{\mathbf{A}}^{y, z} : G_{\mathbf{A}}(y, z) \rightarrow \text{Mor}_{\text{ptls}}(X_{\mathbf{A}}(y), X_{\mathbf{A}}(z))$ is a continuous map, for all $y, z \in \text{Obj}(\mathbf{G}_{\mathbf{A}})$;
- (4) $\tau_{\mathbf{A}}(g \circ h) = \tau_{\mathbf{A}}(g) \circ \tau_{\mathbf{A}}(h)$, and $\tau_{\mathbf{A}}(1_x) = 1_{X_{\mathbf{A}}(x)}$, for all $x, y, z \in \text{Obj}(\mathbf{G}_{\mathbf{A}})$, $g \in G_{\mathbf{A}}(y, z)$ and $h \in G_{\mathbf{A}}(x, y)$.

Let \mathbf{A} and \mathbf{B} be pdp, thus $(f, T) \in \text{Mor}_{\text{pdp}}(\mathbf{A}, \mathbf{B})$ iff

- (1) $f = (f_o, f_m)$ such that $f_o : \text{Obj}(\mathbf{G}_{\mathbf{A}}) \rightarrow \text{Obj}(\mathbf{G}_{\mathbf{B}})$ and $f_m : \text{Mor}_{\mathbf{G}_{\mathbf{A}}} \rightarrow \text{Mor}_{\mathbf{G}_{\mathbf{B}}}$;
- (2) for all $y, z \in \text{Obj}(\mathbf{G}_{\mathbf{A}})$
 - (a) $f_m^{y, z} : G_{\mathbf{A}}(y, z) \rightarrow G_{\mathbf{B}}(f_o(y), f_o(z))$ is a continuous map;
 - (b) $f_m(g \circ h) = f_m(g) \circ f_m(h)$, for all $x \in \text{Obj}(\mathbf{G}_{\mathbf{A}})$, $g \in G_{\mathbf{A}}(y, z)$ and $h \in G_{\mathbf{A}}(x, y)$;
 - (c) $T \in \prod_{x \in \text{Obj}(\mathbf{G}_{\mathbf{A}})} \text{Mor}_{\text{ptls}}(X_{\mathbf{A}}(x), X_{\mathbf{B}}(f_o(x)))$ such that for all $g \in G_{\mathbf{A}}(y, z)$ we have that the following diagram in ptls is commutative

$$\begin{array}{ccc} X_{\mathbf{A}}(y) & \xrightarrow{T(y)} & X_{\mathbf{B}}(f_o(y)) \\ \tau_{\mathbf{A}}(g) \downarrow & & \downarrow \tau_{\mathbf{B}}(f_m(g)) \\ X_{\mathbf{A}}(z) & \xrightarrow{T(z)} & X_{\mathbf{B}}(f_o(z)) \end{array}$$

Under the same line used in the proof of Lemma 1.4.4 we show that

LEMMA 1.4.10. The composition in (16) is a well-defined associative map such that $(f, T) \circ 1_{\mathbf{A}} = 1_{\mathbf{B}} \circ (f, T) = (f, T)$, for all \mathbf{A} and \mathbf{B} pdp and $(f, T) \in \text{Mor}_{\text{pdp}}(\mathbf{A}, \mathbf{B})$.

COROLLARY 1.4.11. There exists a unique \mathcal{U}_0 -type category pdp such that $\text{Obj}(\text{pdp})$ is the set of all \mathcal{U} -type preordered dynamical patterns, for any $\mathbf{A}, \mathbf{B} \in \text{Obj}(\text{pdp})$ the set of morphisms of pdp from \mathbf{A} to \mathbf{B} is the set in (15), and the law of composition of morphisms of pdp is the map

in (16). 1_D defined in Def. 1.4.7 is the unit morphism in \mathfrak{pdp} relative to D , for all $D \in \text{Obj}(\mathfrak{pdp})$. In particular \mathfrak{pdp} is an object of Cat .

PROOF. Since Lemma 1.4.10 and then follows the line of reasoning present in the proof of Cor. 1.4.5, by considering the \mathcal{U} -category ptls instead of tsa . \square

DEFINITION 1.4.12. Let \mathfrak{pdp}_\star be the full subcategory of \mathfrak{pdp} whose object set is the subset of the $A \in \text{Obj}(\mathfrak{pdp})$ such that G_A is a groupoid whose inversion map is continuous.

Note that if $A \in \mathfrak{pdp}_\star$ then $\tau_A(g^{-1}) = \tau_A(g)^{-1}$ for all $g \in G_A$.

DEFINITION 1.4.13. Let \mathfrak{A} be a \mathfrak{dp} , define $\mathfrak{A}^+ := \langle G_{\mathfrak{A}}^{\text{op}}, \sigma_{\mathfrak{A}}^+ \rangle$, where $\sigma_{\mathfrak{A}}^+ := (\mathcal{A}_{\mathfrak{A}}^*, \tau_{\mathfrak{A}}^+)$, with

$$\begin{aligned} \mathcal{A}_{\mathfrak{A}}^* : \text{Obj}(G_{\mathfrak{A}}) &\rightarrow \text{Obj}(\text{ptls}) & x &\mapsto (\mathcal{A}_{\mathfrak{A}}(x))^*, \\ \tau_{\mathfrak{A}}^+ : \text{Mor}_{G_{\mathfrak{A}}^{\text{op}}} &\rightarrow \text{Mor}_{\text{ptls}} & & \\ \tau_{\mathfrak{A}}^+ : \text{Mor}_{G_{\mathfrak{A}}^{\text{op}}}(x, y) &\rightarrow \text{Mor}_{\text{ptls}}(\mathcal{A}_{\mathfrak{A}}^*(x), \mathcal{A}_{\mathfrak{A}}^*(y)) & g &\mapsto (\tau_{\mathfrak{A}}(g))^+, \forall x, y \in G_{\mathfrak{A}}. \end{aligned}$$

Let \mathfrak{B} be a \mathfrak{dp} such that $G_{\mathfrak{B}}$ is a groupoid whose inversion map is continuous. Define $\mathfrak{B}^* := \langle G_{\mathfrak{B}}, \sigma_{\mathfrak{B}}^* \rangle$, where $\sigma_{\mathfrak{B}}^* := (\mathcal{A}_{\mathfrak{B}}^*, \tau_{\mathfrak{B}}^*)$, with $\tau_{\mathfrak{B}}^* : \text{Mor}_{G_{\mathfrak{B}}} \rightarrow \text{Mor}_{\text{ptls}}$ such that

$$\tau_{\mathfrak{B}}^* : \text{Mor}_{G_{\mathfrak{B}}}(x, y) \rightarrow \text{Mor}_{\text{ptls}}(\mathcal{A}_{\mathfrak{B}}^*(x), \mathcal{A}_{\mathfrak{B}}^*(y)) \quad g \mapsto (\tau_{\mathfrak{B}}(g^{-1}))^+, \forall x, y \in G_{\mathfrak{B}}.$$

The previous definition as well the next one are well set since (9), Lemma 1.3.1(2) and Rmk. 1.3.2.

DEFINITION 1.4.14. Let D a category, $\mathbf{a}, \mathbf{b} \in \text{Fct}(D, \text{tsa})$ and $T \in \prod_{d \in D} \text{Mor}_{\text{tsa}}(\mathbf{a}(d), \mathbf{b}(d))$, then define $T^+ \in \prod_{d \in D} \text{Mor}_{\text{ptls}}(\mathbf{b}(d)^*, \mathbf{a}(d)^*)$ such that $T^+(e) := (T(e))^+$, for all $e \in D$.

THEOREM 1.4.15. Let $\mathfrak{A}, \mathfrak{B}$ and \mathfrak{C} be \mathfrak{dp} , \mathfrak{D} be an object of \mathfrak{pdp}_\star , and $(f, T) \in \text{Mor}_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B})$ and $(g, S) \in \text{Mor}_{\mathfrak{dp}}(\mathfrak{B}, \mathfrak{C})$. Thus \mathfrak{A}^+ is a \mathfrak{pdp} which is of \mathcal{U} -type if it is so \mathfrak{A} , and \mathfrak{D}^* is an object of \mathfrak{pdp}_\star . Moreover, $(f, T^+) \in \text{Mor}_{\mathfrak{pdp}}(\mathfrak{B}^+, \mathfrak{A}^+)$ and if we set $(f, T)^+ := (f, T^+)$, then $((g, S) \circ (f, T))^+ = (f, T)^+ \circ (g, S)^+$.

PROOF. Let $\mathfrak{A}, \mathfrak{B}$ be \mathfrak{dp} , \mathfrak{D} be a \mathfrak{dp}_\star , and $(f, T) \in \text{Mor}_{\mathfrak{dp}}(\mathfrak{A}, \mathfrak{B})$. $\tau_{\mathfrak{A}}^+$ and $\tau_{\mathfrak{D}}^*$ are continuous maps since Lemma 1.3.1(1) and Rmk. 1.3.2. Next $\tau_{\mathfrak{A}}^+(h \circ^{\text{op}} g) = \tau_{\mathfrak{A}}^+(h) \circ \tau_{\mathfrak{A}}^+(g)$, and $\tau_{\mathfrak{D}}^*(l \circ m) = \tau_{\mathfrak{D}}^*(l) \circ \tau_{\mathfrak{D}}^*(m)$, since (9), Lemma 1.3.1(2) and Rmk. 1.3.2, where $(\circ)^{\text{op}}$ is the composition of morphisms of $G_{\mathfrak{A}}^{\text{op}}$; hence \mathfrak{A}^+ and \mathfrak{D}^* are \mathfrak{pdp} . Next clearly $f \in \text{Fct}(G_{\mathfrak{B}}^{\text{op}}, G_{\mathfrak{A}}^{\text{op}})$, moreover since the diagram in Rmk. 1.4.3 and (9), Lemma 1.3.1(2) and Rmk. 1.3.2, we deduce that the following diagram in ptls is commutative for all $g \in G_{\mathfrak{B}}^{\text{op}}(z, y)$

$$\begin{array}{ccc} \mathcal{A}_{\mathfrak{A}}^*(f_o(y)) & \xleftarrow{T^+(y)} & \mathcal{A}_{\mathfrak{B}}^*(y) \\ \uparrow \tau_{\mathfrak{A}}^+(f_m(g)) & & \uparrow \tau_{\mathfrak{B}}^+(g) \\ \mathcal{A}_{\mathfrak{A}}^*(f_o(z)) & \xleftarrow{T^+(z)} & \mathcal{A}_{\mathfrak{B}}^*(z) \end{array}$$

Therefore $(f, T^+) \in \text{Mr}_{\text{pdp}}(\mathfrak{B}^+, \mathfrak{A}^+)$ since the diagram in Rmk. 1.4.9. The equality in the statement follows since Lemma 1.3.1(2), (14) and (16). \square

COROLLARY 1.4.16. *The maps $\mathfrak{A} \mapsto \mathfrak{A}^+$, $(f, T) \mapsto (f, T^+)$ and respectively $\mathfrak{A} \mapsto \mathfrak{A}^*$ and $(f, T) \mapsto (f, T^+)$, determine uniquely an element in $\text{Fct}(\text{dp}^{\text{op}}, \text{pdp})$ and respectively in $\text{Fct}(\text{dp}_\star^{\text{op}}, \text{pdp}_\star)$.*

PROOF. By Thm. 1.4.15 and Cor. 1.4.11. \square

Since Prp. 1.3.5(1) and since $\mathfrak{Z}(\mathcal{A}, \mathcal{B})$ depends by \mathcal{A}^* and \mathcal{B}^* rather than by \mathcal{A} and \mathcal{B} , we can state that

PROPOSITION 1.4.17. *There exists a unique subcategory tsa^+ of ptls such that $\text{Obj}(\text{tsa}^+) = \{\mathcal{A}^* \mid \mathcal{A} \in \text{tsa}\}$, and $\text{Mor}_{\text{tsa}^+}(\mathcal{A}^*, \mathcal{B}^*) = \mathfrak{Z}(\mathcal{A}, \mathcal{B})$, for all $\mathcal{A}, \mathcal{B} \in \text{tsa}$.*

Notice that $\mathfrak{A} \in \text{dp}$ implies $\sigma_{\mathfrak{A}}^+ \in \text{Fct}(G_{\mathfrak{A}}^{\text{op}}, \text{tsa}^+)$ and $\mathfrak{D} \in \text{dp}_\star$ implies $\sigma_{\mathfrak{D}}^* \in \text{Fct}(G_{\mathfrak{D}}, \text{tsa}^+)$ since the proof of Thm. 1.4.15, then by recalling Def. 1.3.6 we can set the following

DEFINITION 1.4.18. *Let $\mathfrak{A}, \mathfrak{B}$ and \mathfrak{C} be dp , define*

$$(17) \quad \text{Mor}_{\mathfrak{Chdv}}(\mathfrak{A}, \mathfrak{B}) := \coprod_{f \in \text{Fct}_{\text{top}}(G_{\mathfrak{B}}, G_{\mathfrak{A}})} \text{Mor}_{\text{Fct}(G_{\mathfrak{B}}^{\text{op}}, \text{tsa}^+)}(\sigma_{\mathfrak{B}}^+, \sigma_{\mathfrak{A}}^+ \circ f) \times \text{Mor}_{\text{Fct}(G_{\mathfrak{B}}, \text{ptsa})}(\text{p} \circ \sigma_{\mathfrak{A}} \circ f, \text{p} \circ \sigma_{\mathfrak{B}})$$

and

$$(18) \quad (\circ) : \text{Mor}_{\mathfrak{Chdv}}(\mathfrak{B}, \mathfrak{C}) \times \text{Mor}_{\mathfrak{Chdv}}(\mathfrak{A}, \mathfrak{B}) \rightarrow \text{Mor}_{\mathfrak{Chdv}}(\mathfrak{A}, \mathfrak{C}),$$

$$(g, L, S) \circ (f, H, T) := (f \circ g, (H * 1_g) \circ L, S \circ (T * 1_g)).$$

COROLLARY 1.4.19. *There exists a unique \mathcal{U}_0 -type category \mathfrak{Chdv} such that $\text{Obj}(\mathfrak{Chdv}) = \text{Obj}(\text{dp})$, for any $\mathfrak{A}, \mathfrak{B} \in \text{Obj}(\mathfrak{Chdv})$ the set of morphisms of \mathfrak{Chdv} from \mathfrak{A} to \mathfrak{B} is the set in (17) and the law of composition of morphisms of \mathfrak{Chdv} is the map in (18), moreover $(1_{G_{\mathfrak{A}}}, 1_{\sigma_{\mathfrak{A}}^+}, 1_{\text{p} \circ \sigma_{\mathfrak{A}}})$ is the unit morphism in \mathfrak{Chdv} relative to \mathfrak{A} . In particular \mathfrak{Chdv} is an object of Cat .*

PROOF. By Cor. 1.4.5 and Cor. 1.4.11, and since $\text{Obj}(\text{Fct}(A, B)) = \text{Obj}(\text{Fct}(A^{\text{op}}, B^{\text{op}}))$ for all categories A, B . \square

THEOREM 1.4.20. *The maps $\mathfrak{A} \mapsto \mathfrak{A}$ and $(f, T) \mapsto (f, T^+, T)$ determine uniquely an element $\Psi \in \text{Fct}(\text{dp}, \mathfrak{Chdv})$.*

PROOF. Since Cor. 1.4.16. \square

CONVENTION 1.4.21. For any map U on a set X and into $\text{Mor}_{\mathfrak{Chdv}}$ let U_j and U_1^o, U_1^m , $j \in \{1, 2, 3\}$ be maps on X such that $U(M) = (U_1(M), U_2(M), U_3(M))$ and $U_1(M) = (U_1^o(M), U_1^m(M))$, namely $U_1^o(M) = (U_1(M))_o$ is the object map and $U_1^m(M) = (U_1(M))_m$ is the morphism map respectively of the functor $U_1(M)$ for all $M \in X$. Moreover $U_3^+(M)(\cdot) := U_3(M)(\cdot)^+$, where we recall that $U_3(M)$ and $U_2(M)$ are natural transformations. If $U = \mathbf{a}_m$ with \mathbf{a} any functor at values in \mathfrak{Chdv} , then we convey to denote $(\mathbf{a}_m)_1^o$ and $(\mathbf{a}_m)_1^m$ simply by \mathbf{a}_1^o and \mathbf{a}_1^m , while $(\mathbf{a}_m)_2$ and $(\mathbf{a}_m)_3$ by \mathbf{a}_2 and \mathbf{a}_3 respectively. We apply to objects of \mathfrak{Chdv} the notations in Def. 1.4.1 for objects of dp .

Next we shall introduce the concept of trajectory associated with any species of dynamical patterns, namely a functor from a category \mathfrak{D} said context category and $\mathfrak{Ch}\mathfrak{d}\mathfrak{v}$, and any context.

DEFINITION 1.4.22 (Trajectories). *Let \mathfrak{D} be a category, $\mathbf{a} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Ch}\mathfrak{d}\mathfrak{v})$ and $M \in \mathfrak{D}$ define*

$$\mathfrak{f}^{\mathbf{a}, M} \in \prod_{(x, y) \in G_{\mathbf{a}(M)} \times G_{\mathbf{a}(M)}} \text{Mor}_{\text{set}}(\mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(y)} \times \mathcal{A}_{\mathbf{a}(M)}(x)_{ob}, \mathbb{R}^{G_{\mathbf{a}(M)}(x, y)}),$$

such that for all $x, y \in G_{\mathbf{a}(M)}$, $(\psi, A) \in \mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(y)} \times \mathcal{A}_{\mathbf{a}(M)}(x)_{ob}$ and $g \in G_{\mathbf{a}(M)}(x, y)$

$$\mathfrak{f}_{(\psi, A)}^{\mathbf{a}, M, x, y}(g) \doteq \psi(\tau_{\mathbf{a}(M)}(g)A),$$

set

$$\text{Tr}^{\mathbf{a}}(M, x, y) := \{\mathfrak{f}_{(\psi, A)}^{\mathbf{a}, M, x, y} \mid (\psi, A) \in \mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(y)} \times \mathcal{A}_{\mathbf{a}(M)}(x)_{ob}\}.$$

Let $\mathbf{b} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Ch}\mathfrak{d}\mathfrak{v})$, $N \in \mathfrak{D}$, $t \in G_{\mathbf{a}(M)}$, and $u, v \in G_{\mathbf{b}(N)}$, define

$$\mathfrak{d}^{\mathbf{a}, M; \mathbf{b}, N; t, u, v} : \mathfrak{Z}(\mathcal{A}_{\mathbf{a}(M)}(t), \mathcal{A}_{\mathbf{b}(N)}(v)) \times \mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(t)}^{\mathfrak{h}} \times \text{Ef}(\mathcal{A}_{\mathbf{b}(N)}(u)) \rightarrow [0, 1]^{G_{\mathbf{b}(N)}(u, v)},$$

$$(J, \omega, e) \mapsto \left(l \mapsto \frac{J(\omega)(\tau_{\mathbf{b}(N)}(l)e)}{\omega(1)} \right).$$

We let $\mathfrak{d}^{\mathbf{b}, N; u, v}$ denote $\mathfrak{d}^{\mathbf{a}, M; \mathbf{b}, N; t, u, v}$ whenever it will be clear by the context the functor \mathbf{a} , and M and t involved.

Notice that $\omega(1)\mathfrak{d}_{(J, \omega, e)}^{\mathbf{b}, N; u, v}(l) = \mathfrak{f}_{(J(\omega), e)}^{\mathbf{b}, N; u, v}(l)$. Although the precise physical interpretation will follow after Def. 1.4.29, we can say that $\mathfrak{f}_{(\psi, A)}^{\mathbf{a}, M, x, y}$ is roughly the trajectory mapping any morphism g of the dynamical category of the dynamical pattern $\mathbf{a}(M)$ - associated with the context M and implemented by the species \mathbf{a} - into the measure in the statistical ensemble ψ of the observable $\tau_{\mathbf{a}(M)}(g)A$. Next we give a variant of the above definition suitable to better reveal the observable dependency.

DEFINITION 1.4.23 (Observable trajectories). *Let \mathfrak{D} be a category, $\mathbf{a} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Ch}\mathfrak{d}\mathfrak{v})$ and $M \in \mathfrak{D}$ define*

$$\mathfrak{t}^{\mathbf{a}, M} \in \prod_{(x, y) \in G_{\mathbf{a}(M)} \times G_{\mathbf{a}(M)}} \text{Mor}_{\text{set}}(\mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(y)}, \text{Mor}_{\text{set}}(\mathcal{A}_{\mathbf{a}(M)}(x)_{ob}, \mathbb{R}^{G_{\mathbf{a}(M)}(x, y)})),$$

$$\mathfrak{t}^{\mathbf{a}, M, x, y, \psi}(A) := \mathfrak{f}_{(\psi, A)}^{\mathbf{a}, M, x, y}.$$

Set

$$\mathfrak{O}^{\mathbf{a}}(M) := \{\mathfrak{t}^{\mathbf{a}, M, x, y, \psi} \mid x, y \in G_{\mathbf{a}(M)}, \psi \in \mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(y)}\}.$$

$\mathfrak{f}^{\mathbf{a}, M, x, y}$ is defined over all the couples of statistical ensembles and observables, however when dealing with a precise experimental setting one encounters subsets of those couples which are in general equivariant with respect to geometrical and dynamical transformations implemented by the species \mathbf{a} . Thus we introduce the following

DEFINITION 1.4.24 (Experimental settings). Let \mathfrak{D} be a category and $\mathbf{a} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, let $\text{Exp}(\mathbf{a})$ be called the set of experimental settings associated with \mathbf{a} , and defined to be the set of the $\mathfrak{E} = (\mathfrak{S}, \mathbf{R})$ such that for all $M \in \mathfrak{D}$ the following holds

- (1) \mathbf{R}_M is a subcategory of $\mathbf{G}_{\mathbf{a}(M)}$;
- (2) $\mathfrak{S}_M \in \prod_{t \in \mathbf{R}_M} \mathcal{P}(\mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(t)})$;
- (3) for all $N \in \mathfrak{D}$ and $\phi \in \text{Mor}_{\mathfrak{D}}(M, N)$
 - (a) $\mathbf{a}_1(\phi) \in \mathbf{Fct}_{\text{top}}(\mathbf{R}_N, \mathbf{R}_M)$,
 - (b) $\mathbf{a}_3^+(\phi)(u)\mathfrak{S}_N(u) \subseteq \mathfrak{S}_M(\mathbf{a}_1^o(\phi)u)$, for all $u \in \mathbf{R}_N$;
- (4) $\tau_{\mathbf{a}(M)}^+(g)\mathfrak{S}_M(z) \subseteq \mathfrak{S}_M(y)$, for all $y, z \in \mathbf{R}_M$ and $g \in \mathbf{R}_M(y, z)$.

If in addition (3b) holds by replacing \mathbf{a}_3^+ by \mathbf{a}_2 , then $(\mathfrak{S}, \mathbf{R})$ is said to be complete. Moreover let $\text{Exp}^*(\mathbf{a})$ be the subset of the $(\mathfrak{S}, \mathbf{R})$ in $\text{Exp}(\mathbf{a})$ s.t. for all $M \in \mathfrak{D}$, $z \in \mathbf{R}_M$ and $a \in \mathcal{A}_{\mathbf{a}(M)}(z)$

$$(19) \quad \gamma^+(a)\mathfrak{S}_M(z) \subseteq \mathfrak{S}_M(z).$$

DEFINITION 1.4.25. Let \mathfrak{D} be a category and $\mathbf{a} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, define $\mathfrak{P}^{\mathbf{a}}$ and $\mathbf{G}^{\mathbf{a}}$ be maps on $\text{Obj}(\mathfrak{D})$ such that $\mathbf{G}_M^{\mathbf{a}} := \mathbf{G}_{\mathbf{a}(M)}$ for all $M \in \mathfrak{D}$ and $\mathfrak{P}_M^{\mathbf{a}}(t) := \mathfrak{P}_{\mathcal{A}_{\mathbf{a}(M)}(t)}$ for all $t \in \mathbf{G}_{\mathbf{a}(M)}$. It is easy to see that $(\mathfrak{P}^{\mathbf{a}}, \mathbf{G}^{\mathbf{a}}) \in \text{Exp}^*(\mathbf{a})$ and it is complete, we call $(\mathfrak{P}^{\mathbf{a}}, \mathbf{G}^{\mathbf{a}})$ the standard experimental setting associated with \mathbf{a} .

REMARK 1.4.26. Let $\mathbf{S}, \mathbf{T} \in \text{Mor}_{\mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})}$ such that $d(\mathbf{T}) = c(\mathbf{S})$, $M \in \mathfrak{D}$ and let $\mathbf{c} = c(\mathbf{T})$. Thus by Conv. 1.4.21

$$\begin{aligned} (\mathbf{T} \circ \mathbf{S})(M) &= ((\mathbf{T} \circ \mathbf{S})_1(M), (\mathbf{T} \circ \mathbf{S})_2(M), (\mathbf{T} \circ \mathbf{S})_3(M)) \\ &= (\mathbf{S}_1(M) \circ \mathbf{T}_1(M), (\mathbf{S}_2(M) * \mathbf{1}_{\mathbf{T}_1(M)}) \circ \mathbf{T}_2(M), \mathbf{T}_3(M) \circ (\mathbf{S}_3(M) * \mathbf{1}_{\mathbf{T}_1(M)})), \end{aligned}$$

where the second equality arises since $(\mathbf{T} \circ \mathbf{S})(M) = \mathbf{T}(M) \circ \mathbf{S}(M)$ and by the definition of morphism composition in \mathfrak{Chdv} . In particular by (2) we obtain for all $y \in \text{Obj}(\mathbf{G}_M^{\mathbf{c}})$

$$(\mathbf{T} \circ \mathbf{S})_3(M)(y) = \mathbf{T}_3(M)(y) \circ \mathbf{S}_3(M)(\mathbf{T}_1^o(M)y),$$

and $(\mathbf{T} \circ \mathbf{S})_3^+(M)(y) = ((\mathbf{T} \circ \mathbf{S})_3(M)(y))^+$.

Next we introduce the auxiliary concept of link in order to express the basic principle of equiformity in Prp. 1.4.34. Roughly a link between experimental settings in general of different species, provides a functor between the respective dynamical categories, a covariant map between the respective collections of statistical ensembles and equivariant maps between composition of dynamical and geometrical transformations of the respective species.

DEFINITION 1.4.27 (Links). Let \mathfrak{D} be a category, $\mathbf{a}, \mathbf{b} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, and $(\mathfrak{S}^{\mathbf{a}}, \mathbf{R}^{\mathbf{a}}) \in \text{Exp}(\mathbf{a})$, $(\mathfrak{S}^{\mathbf{b}}, \mathbf{R}^{\mathbf{b}}) \in \text{Exp}(\mathbf{b})$. We define \mathbf{T} to be a link from $(\mathfrak{S}^{\mathbf{b}}, \mathbf{R}^{\mathbf{b}})$ to $(\mathfrak{S}^{\mathbf{a}}, \mathbf{R}^{\mathbf{a}})$ if $\mathbf{T} \in \prod_{O \in \mathfrak{D}} \text{Mor}_{\mathfrak{Chdv}}(\mathbf{a}(O), \mathbf{b}(O))$ such that for all $M, N \in \mathfrak{D}$, $\phi \in \text{Mor}_{\mathfrak{D}}(M, N)$, $y, z \in \mathbf{R}_N^{\mathbf{b}}$ and $g \in \mathbf{R}_N^{\mathbf{b}}(y, z)$ we have

- (1) $\mathbf{T}_1(N) \in \mathbf{Fct}_{\text{top}}(\mathbf{R}_N^{\mathbf{b}}, \mathbf{R}_N^{\mathbf{a}})$;
- (2) $\mathbf{T}_3^+(N)(z)\mathfrak{S}_N^{\mathbf{b}}(z) \subseteq \mathfrak{S}_N^{\mathbf{a}}(\mathbf{T}_1^o(N)z)$;
- (3) $\mathbf{T}_3(N)(z) \circ \tau_{\mathbf{a}(N)}(\mathbf{T}_1^m(N)g) \circ \mathbf{a}_3(\phi)(\mathbf{T}_1^o(N)y) = \mathbf{b}_3(\phi)(z) \circ \tau_{\mathbf{b}(M)}(\mathbf{b}_1^m(\phi)g) \circ \mathbf{T}_3(M)(\mathbf{b}_1^o(\phi)y)$;

$$(4) \quad T_2(M)(b_1^o(\phi)y) \circ \tau_{b(M)}^+(b_1^m(\phi)g) \circ b_2(\phi)(z) = a_2(\phi)(T_1^o(N)y) \circ \tau_{a(N)}^+(T_1^m(N)g) \circ T_2(N)(z).$$

If in addition (2) holds by replacing T_3^+ by T_2 , then T is said to be complete.

CONVENTION 1.4.28. For any category \mathfrak{D} , $a, b \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, and $T \in \prod_{O \in \mathfrak{D}} \mathbf{Mor}_{\mathfrak{Chdv}}(a(O), b(O))$ by abuse of language we generalize the notation for natural transformations so that $d(T) = a$ and $c(T) = b$ and call them domain and codomain of T respectively.

Let us introduce the interpretation of the data above defined.

DEFINITION 1.4.29. We call (s, u) a semantics for \mathfrak{Chdv} , if it extends a semantics Def. 1.3.8 in the following way. For any category \mathfrak{D} , any $a, b, c \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, $T \in \prod_{O \in \mathfrak{D}} \mathbf{Mor}_{\mathfrak{Chdv}}(a(O), b(O))$, $S \in \prod_{O \in \mathfrak{D}} \mathbf{Mor}_{\mathfrak{Chdv}}(b(O), c(O))$ and $\mathfrak{E} = (\mathfrak{S}, R) \in \mathbf{Exp}(b)$, we have for all $M, N, O \in \mathbf{Obj}(\mathfrak{D})$, $\phi \in \mathbf{Mor}_{\mathfrak{D}}(M, N)$, $\psi \in \mathbf{Mor}_{\mathfrak{D}}(N, O)$, $x, y \in G_a^M$, $h, g \in G_a^M(x, y)$, $u \in \mathbf{Mor}_{G_a^N}$, $n \in G_a^M$, $z \in G_b^M$, $w \in \mathbf{Mor}_{G_b^M}$ and $r \in R_M$

- (1) $u(1_N) \equiv$ producing no variations;
- (2) $s(x) \equiv$ the region $u(x)$;
- (3) $s(g) \equiv$ the action $u(g)$ mapping $s(x)$ into $s(y)$;
- (4) $s(hg) \equiv s(h)$ after $s(g)$;
- (5) $s(\phi) \equiv$ the geometric transformation $u(\phi)$;
- (6) $s(\phi) \equiv$ the reference frame $u(\phi)^9$;
- (7) $s(\psi \circ \phi) \equiv s(\psi)$ after $s(\phi)$;
- (8) $s(a) \equiv$ the species $u(a)$;
- (9) $s(d(a)) \equiv$ the environment domain of $s(a)$;
- (10) $s(M) \equiv$ the context $u(M)$;
- (11) $u(c(\phi)) \equiv$ projection of $s(d(\phi))$ through $s(\phi)$;
- (12) $u(c(\phi)) \equiv$ obtained by transforming $s(d(\phi))$ through $s(\phi)$;
- (13) $s(a(M)) \equiv$ dynamical pattern associated with $s(M)$ and implemented by $s(a)$;
- (14) $s(a_1(\phi)) \equiv$ the probe $u(a_1(\phi))$;
- (15) $u(a_1(\phi)) \equiv$ assembled by $s(a)$ and implementing $s(\phi)$;
- (16) $u(a_1^o(\phi)x) \equiv$ resulting next $s(a_1(\phi))$ applies on $s(x)$
- (17) $u(a_1^m(\phi)(u)) \equiv$ that, via $s(a_1(\phi))$, operates in $s(a(M))$ as $s(u)$ operates in $s(a(N))$;
- (18) $u(a_1^m(\phi)(u)) \equiv$ resulting next $s(a_1(\phi))$ applies on $s(u)$;
- (19) $u(a_2(\phi)) \equiv u(a_1(\phi))$;
- (20) $u(a_2(\phi)(x)) \equiv$ placed in $s(x)$, $u(a_2(\phi))$;
- (21) $u(a_3(\phi)) \equiv u(a_1(\phi))$;
- (22) $u(a_3(\phi)(x)) \equiv$ placed in $s(x)$, $u(a_3(\phi))$;
- (23) $s(T) \equiv$ the connector $u(T)$;
- (24) $u(T) \equiv$ adding the charge T^{10} ;
- (25) $u(T) \equiv$ from $s(d(T))$ to $s(c(T))$;

⁹to be understood in the broad sense of passive transformation.

¹⁰The reason of such a choice is related to Thm. 1.4.46(1).

- (26) $\mathfrak{s}(\mathbf{T}) \equiv$ the sector $\mathfrak{u}(\mathbf{T})$, if $d(\mathbf{T}) = c(\mathbf{T})$;
- (27) $\mathfrak{u}(\mathbf{T}) \equiv$ of $\mathfrak{s}(d(\mathbf{T}))$, if $d(\mathbf{T}) = c(\mathbf{T})$;
- (28) $\mathfrak{u}(\mathbf{1}_a) \equiv$ canonically associated with $\mathfrak{s}(\mathbf{a})$;
- (29) $\mathfrak{s}(\mathbf{T} \circ \mathbf{S}) \equiv \mathfrak{s}(\mathbf{T})$ following $\mathfrak{s}(\mathbf{S})$;
- (30) $\mathfrak{s}(\mathbf{T}_1(\mathbf{M})) \equiv$ the probe $\mathfrak{u}(\mathbf{T}_1(\mathbf{M}))$;
- (31) $\mathfrak{u}(\mathbf{T}_1(\mathbf{M})) \equiv$ assembled by $\mathfrak{s}(\mathbf{T})$ and transporting into $\mathfrak{s}(\mathbf{a}(\mathbf{M}))$;
- (32) $\mathfrak{u}(\mathbf{T}_1(\mathbf{M})^o(z)) \equiv$ resulting next $\mathfrak{s}(\mathbf{T}_1(\mathbf{M}))$ applies on $\mathfrak{s}(z)$;
- (33) $\mathfrak{u}(\mathbf{T}_1(\mathbf{M})^m(w)) \equiv$ that, via $\mathfrak{s}(\mathbf{T}_1(\mathbf{M}))$, operates in $\mathfrak{s}(\mathbf{a}(\mathbf{M}))$ as $\mathfrak{s}(w)$ operates in $\mathfrak{s}(\mathbf{b}(\mathbf{M}))$;
- (34) $\mathfrak{u}(\mathbf{T}_1(\mathbf{M})^m(w)) \equiv$ resulting next $\mathfrak{s}(\mathbf{T}_1(\mathbf{M}))$ applies on $\mathfrak{s}(w)$;
- (35) $\mathfrak{u}(\mathbf{T}_2(\mathbf{M})) \equiv \mathfrak{u}(\mathbf{T}_1(\mathbf{M}))$;
- (36) $\mathfrak{u}(\mathbf{T}_2(\mathbf{M})(z)) \equiv$ placed in $\mathfrak{s}(z)$, $\mathfrak{s}(\mathbf{T}_2(\mathbf{M}))$;
- (37) $\mathfrak{u}(\mathbf{T}_3^+(\mathbf{M})) \equiv \mathfrak{u}(\mathbf{T}_1(\mathbf{M}))$;
- (38) $\mathfrak{u}(\mathbf{T}_3^+(\mathbf{M})(z)) \equiv$ placed in $\mathfrak{s}(z)$, $\mathfrak{u}(\mathbf{T}_3^+(\mathbf{M}))$;
- (39) $\mathfrak{u}(\mathbf{T}_3(\mathbf{M})) \equiv$ assembled by $\mathfrak{s}(\mathbf{T})$ and transporting into $\mathfrak{s}(\mathbf{b}(\mathbf{M}))$;
- (40) $\mathfrak{u}(\mathbf{T}_3(\mathbf{M})(n)) \equiv$ placed in $\mathfrak{s}(n)$, $\mathfrak{u}(\mathbf{T}_3(\mathbf{M}))$;
- (41) $\mathfrak{u}(\tau_{\mathbf{a}(\mathbf{M})}(g)) \equiv$ assembled by $\mathfrak{s}(\mathbf{a})$ and implementing in $\mathfrak{s}(\mathbf{M})$ $\mathfrak{s}(g)$;
- (42) $\mathfrak{u}(\tau_{\mathbf{a}(\mathbf{M})}(g)) \equiv$ assembled by $\mathfrak{s}(\mathbf{a})$ and implementing $\mathfrak{s}(g)$ in $\mathfrak{s}(\mathbf{M})$;
- (43) $\mathfrak{s}(\mathfrak{S}) \equiv$ the bundle \mathfrak{S} of empirical sectors of $\mathfrak{s}(\mathbf{b})$;
- (44) $\mathfrak{s}(\mathbf{R}) \equiv$ the bundle \mathbf{R} of dynamical categories of $\mathfrak{s}(\mathbf{b})$;
- (45) $\mathfrak{s}(\mathfrak{S}_M(r)) \equiv$ the fiber $\mathfrak{u}(\mathfrak{S}_M(r))$;
- (46) $\mathfrak{u}(\mathfrak{S}_M(r)) \equiv$ in $\mathfrak{s}(r)$ of $\mathfrak{s}(\mathbf{M})$ of $\mathfrak{s}(\mathfrak{S})$;
- (47) $\mathfrak{s}(\mathbf{R}_M) \equiv$ the fiber $\mathfrak{u}(\mathbf{R}_M)$;
- (48) $\mathfrak{u}(\mathbf{R}_M) \equiv$ in $\mathfrak{s}(\mathbf{M})$ of $\mathfrak{s}(\mathbf{R})$;
- (49) $\mathfrak{s}(\mathfrak{E}) \equiv$ the experimental setting $\mathfrak{u}(\mathfrak{E})$;
- (50) $\mathfrak{u}(\mathfrak{E}) \equiv$ determined by $\mathfrak{s}(\mathfrak{S}_Q(a))$ and $\mathfrak{s}(\mathbf{R}_Q)$, for all $Q \in \mathfrak{D}$ and $a \in \mathbf{R}_Q$.

Moreover for all sets A, B and maps f on A we require

- (1) $\mathfrak{s}(A) \equiv$ the collection $\mathfrak{u}(A)$;
- (2) $\mathfrak{u}(A) \equiv$ of $\mathfrak{s}(t)$ such that t belongs to A ;
- (3) $\mathfrak{s}(f(A)) \equiv$ resulting next $\mathfrak{s}(f)$ interacts with $\mathfrak{s}(A)$;
- (4) $\mathfrak{s}(A = B) \equiv \mathfrak{s}(A)$ equals $\mathfrak{s}(B)$;
- (5) $\mathfrak{s}(A \subseteq B) \equiv \mathfrak{s}(A)$ is contained in $\mathfrak{s}(B)$;
- (6) $\mathfrak{s}(A \supseteq B) \equiv \mathfrak{s}(A)$ contains $\mathfrak{s}(B)$,

and for any $\mathfrak{Q} \in \text{Exp}(\mathbf{a})$ let $\mathfrak{s}(\mathfrak{Q})$ be the collection of sentences obtained by applying the semantics ($\mathfrak{s}\mathfrak{u}$) to the inclusions inheriting Def. 1.4.24(3a) and the inclusions stated in Def. 1.4.24(3b,4). Here we recall that $\mathbf{1}_N$ is the unit morphism in \mathfrak{D} with domain N , and $\mathbf{1}_b$ is the unit morphism in $\mathfrak{Chd}\mathfrak{v}$ relative to \mathbf{b} . Moreover for any $\omega \in \mathfrak{P}_{\mathcal{A}_{\mathbf{a}(\mathbf{M})(x)}}$, $A \in \mathcal{A}_{\mathbf{a}(\mathbf{M})(x)_{ob}}$, $\mathfrak{h} \in \text{Tr}^{\mathbf{a}}(\mathbf{M}, x, y)$, $H \subseteq G_{\mathbf{a}^M}(x, y)$, map F whose range is a subset of H and $\text{dom}(F) \subseteq G_{\mathbf{b}}^N(k, l)$ with $k, l \in G_{\mathbf{b}}^N$, $K \subseteq \text{dom}(F)$, and $g \in H$ we have

- (1) $\mathfrak{s}(\mathfrak{h}) \equiv$ the trajectory $\mathfrak{u}(\mathfrak{h})$;
- (2) $\mathfrak{u}(\mathfrak{f}_{(\omega, A)}^{\mathbf{a}, M, x, y}) \equiv$ of $\mathfrak{s}(\mathbf{a})$ in $\mathfrak{s}(\mathbf{M})$, defined on the set of actions mapping $\mathfrak{s}(x)$ into $\mathfrak{s}(y)$, and whose initial conditions are $\mathfrak{s}(\omega)$ and $\mathfrak{s}(A)$;

- (3) $s(\bar{f}^{a,M}) \equiv$ the trajectory of $s(a)$ in $s(M)$;
- (4) $s(h \upharpoonright H) \equiv s(h)$ restricted to H ;
- (5) $u(F^+(h \upharpoonright H)) \equiv$ that via $s(F)$ appears in $s(b(N))$ as $s(h \upharpoonright H)$ appears in $s(a(M))$;
- (6) $u(F^+(h \upharpoonright H)) \equiv$ resulting next $s(F)$ applies on $s(h \upharpoonright H)$;
- (7) $s(F \upharpoonright K) \equiv s(F)$, restricted to K ;
- (8) $(h \upharpoonright H)(g)$ is the value at $s(g)$ of $s(h \upharpoonright H)$.

DEFINITION 1.4.30 (Active and passive interpretations). Under the notation in Def. 1.4.29, let us define

- (1) Active interpretation.
 - (a) $s(\phi) \equiv$ the action of $u(\phi)$;
 - (b) $u(c(\phi)) \equiv$ obtained by transforming $s(d(\phi))$ through $s(\phi)$;
 - (c) $u(F^+(h \upharpoonright H)) \equiv$ resulting next $s(F)$ applies on $s(h \upharpoonright H)$.
- (2) Passive interpretation.
 - (a) $s(\phi) \equiv$ the reference frame $u(\phi)$;
 - (b) $u(c(\phi)) \equiv$ projection of $s(d(\phi))$ through $s(\phi)$;
 - (c) $u(F^+(h \upharpoonright H)) \equiv$ that via $s(F)$ appears in $s(b(N))$ as $s(h \upharpoonright H)$ appears in $s(a(M))$.

In the passive interpretation we have for all $\theta \in \text{Mor}_{\mathfrak{D}}(Q, M)$ and $B \in \mathcal{A}_{a(Q)}(a_1^o(\theta)x)_{ob}$ that $s(a_3(\theta)(x)B) \equiv$ the observable resulting next the observable $u(B)$ interacts with the device placed in $s(x)$, assembled by the species $u(a)$ and implementing the reference frame $u(\theta)$. Similarly in the case of statistical ensembles but notice by replacing any device with the channel obtained by conjugation which acts contravariantly. Thus we convey to adopt the following

CONVENTION 1.4.31. Under the notation in Def. 1.4.29 for all $\theta \in \text{Mor}_{\mathfrak{D}}(Q, M)$, $B \in \mathcal{A}_{a(Q)}(a_1^o(\theta)x)_{ob}$ and $\psi \in \mathfrak{P}_{\mathcal{A}_{a(N)}(x)}$ in the passive interpretation we set

- (1) $s(a_3(\theta)(x)B) \equiv$ the observable $u(B)$ as detected in $s(x)$ in the reference frame $u(\theta)$ assembled by the species $u(a)$;
- (2) $s(a_3^+(\phi)(x)\psi) \equiv$ the statistical ensemble that would appear as $s(\psi)$, whenever detected in $s(x)$ in the reference frame $u(\phi)$ assembled by the species $u(a)$.

REMARK 1.4.32 (Passive interpretation applied to trajectories). Under the notation in Def. 1.4.29 we obtain in the passive interpretation what follows for all $\theta \in \text{Mor}_{\mathfrak{D}}(Q, M)$ and $B \in \mathcal{A}_{a(Q)}(a_1^o(\theta)x)_{ob}$

- (1) $s(c(\phi)) \equiv$ the context projection of the context $u(d(\phi))$ through the reference frame $u(\phi)$;
- (2) $s(\bar{f}_{(\omega, a_3(\theta)(x)B)}^{a,M,x,y}) \equiv$ the trajectory of species a in the context projection of the context $u(Q)$ through the reference frame $u(\theta)$, defined on the set of actions mapping $s(x)$ into $s(y)$, and whose initial conditions are $s(\omega)$ and the observable $u(B)$ as detected in $s(x)$ in the reference frame $u(\theta)$ assembled by the species $u(a)$.

CONVENTION 1.4.33. For the remaining of the paper let (s, u) be a fixed semantics for \mathfrak{Chdv} .

Prp. 1.4.34 is a straightforward consequence of Def. 1.4.27(3) while Prp. 1.4.37 and Prp. 1.4.40 follow by Prp. 1.4.34.

PROPOSITION 1.4.34 (Equiformity principle invariant form). *Let \mathfrak{D} be a category, $\mathbf{a}, \mathbf{b} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Ch}\mathfrak{d}\mathfrak{v})$, and $\mathfrak{E}^{\mathbf{a}} = (\mathfrak{S}^{\mathbf{a}}, \mathbf{R}^{\mathbf{a}}) \in \mathbf{Exp}(\mathbf{a})$, $\mathfrak{E}^{\mathbf{b}} = (\mathfrak{S}^{\mathbf{b}}, \mathbf{R}^{\mathbf{b}}) \in \mathbf{Exp}(\mathbf{b})$, and \mathbf{T} be a link from $\mathfrak{E}^{\mathbf{b}}$ to $\mathfrak{E}^{\mathbf{a}}$. Thus for all $M, N \in \mathfrak{D}$, $\phi \in \mathbf{Mor}_{\mathfrak{D}}(M, N)$, $y, z \in \mathbf{R}_N^{\mathbf{b}}$ and $g \in \mathbf{R}_N^{\mathbf{b}}(y, z)$ we have for all $\psi \in \mathfrak{S}_N^{\mathbf{b}}(z)$ and $A \in \mathcal{A}_{\mathbf{a}(M)}((\mathbf{T}_1^{\mathbf{o}}(M) \circ \mathbf{b}_1^{\mathbf{o}}(\phi))y)_{ob}$*

$$\mathbf{f}_{(\mathbf{T}_3^{\dagger}(N)(z)\psi, \mathbf{a}_3(\phi)(\mathbf{T}_1^{\mathbf{o}}(N)y)A)}^{\mathbf{a}, N, \mathbf{T}_1^{\mathbf{o}}(N)y, \mathbf{T}_1^{\mathbf{o}}(N)z}(\mathbf{T}_1^m(N)g) = \mathbf{f}_{(\mathbf{b}_3^{\dagger}(\phi)(z)\psi, \mathbf{T}_3(M)(\mathbf{b}_1^{\mathbf{o}}(\phi)y)A)}^{\mathbf{b}, M, \mathbf{b}_1^{\mathbf{o}}(\phi)y, \mathbf{b}_1^{\mathbf{o}}(\phi)z}(\mathbf{b}_1^m(\phi)g).$$

By employing Def. 1.3.8 and Def. 1.4.29 we translate the above principle as follows.

PROPOSITION 1.4.35 (Interpretation of the equiformity principle). *Under the hypothesis of Prp. 1.4.34 we have what follows. The measure in the statistical ensemble $\mathbf{u}(\mathbf{T}_3^{\dagger}(N)(z)\psi)$ of the observable resulting next the device $\mathbf{u}(\tau_{\mathbf{a}(N)}(\mathbf{T}_1^m(N)g))$ interacts with the observable $\mathbf{u}(\mathbf{a}_3(\phi)(\mathbf{T}_1^{\mathbf{o}}(N)y)A)$, equals the measure in the statistical ensemble $\mathbf{u}(\mathbf{b}_3^{\dagger}(\phi)(z)\psi)$ of the observable resulting next the device $\mathbf{u}(\tau_{\mathbf{b}(M)}(\mathbf{b}_1^m(\phi)g))$ interacts with the observable $\mathbf{u}(\mathbf{T}_3(M)(\mathbf{b}_1^{\mathbf{o}}(\phi)y)A)$. Where*

- $\mathbf{u}(\mathbf{T}_3^{\dagger}(N)(z)\psi) \equiv$ resulting next any sample of the statistical ensemble $\mathbf{u}(\psi)$ interacts with the channel placed in the region $\mathbf{u}(z)$, assembled by the connector adding the charge \mathbf{T} and transporting into $\mathbf{s}(\mathbf{a}(N))$;
- $\mathbf{u}(\mathbf{T}_3^{\dagger}(N)(z)\psi) \equiv$ resulting next any sample of the statistical ensemble $\mathbf{u}(\psi)$ interacts with the channel induced by the device placed in the region $\mathbf{u}(z)$, assembled by the connector adding the charge \mathbf{T} and transporting into $\mathbf{s}(\mathbf{b}(N))$;
- $\mathbf{u}(\tau_{\mathbf{a}(N)}(\mathbf{T}_1^m(N)g)) \equiv$ assembled by the species $\mathbf{u}(\mathbf{a})$ and implementing in the context $\mathbf{u}(N)$ the action $\mathbf{u}(\mathbf{T}_1^m(N)g)$ mapping the region $\mathbf{u}(\mathbf{T}_1^{\mathbf{o}}(N)y)$ into the region $\mathbf{u}(\mathbf{T}_1^{\mathbf{o}}(N)z)$
- $\mathbf{u}(\mathbf{T}_1^m(N)g) \equiv$ resulting next the probe, assembled by the connector adding the charge \mathbf{T} and transporting into $\mathbf{s}(\mathbf{a}(N))$, applies on the action $\mathbf{u}(g)$ mapping the region $\mathbf{u}(y)$ into the region $\mathbf{u}(z)$;
- $\mathbf{u}(\mathbf{a}_3(\phi)(\mathbf{T}_1^{\mathbf{o}}(N)y)A) \equiv$ resulting next the observable $\mathbf{u}(A)$ interacts with the device placed in the region $\mathbf{u}(\mathbf{T}_1^{\mathbf{o}}(N)y)$ assembled by the species $\mathbf{u}(\mathbf{a})$ and implementing the geometric transformation $\mathbf{u}(\phi)$;
- $\mathbf{u}(\mathbf{T}_1^{\mathbf{o}}(N)y) \equiv$ resulting next the probe, assembled by the connector adding the charge \mathbf{T} and transporting into $\mathbf{s}(\mathbf{a}(N))$, applies on the region $\mathbf{u}(y)$;
- $\mathbf{u}(\mathbf{b}_3^{\dagger}(\phi)(z)\psi) \equiv$ resulting next any sample of the statistical ensemble $\mathbf{u}(\psi)$ interacts with the channel induced by the device placed in the region $\mathbf{u}(z)$, assembled by the species $\mathbf{u}(\mathbf{b})$ and implementing the geometric transformation $\mathbf{u}(\phi)$;
- $\mathbf{u}(\tau_{\mathbf{b}(M)}(\mathbf{b}_1^m(\phi)g)) \equiv$ assembled by the species $\mathbf{u}(\mathbf{b})$ and implementing in the context $\mathbf{u}(M)$ the action $\mathbf{u}(\mathbf{b}_1^m(\phi)g)$ mapping the region $\mathbf{u}(\mathbf{b}_1^{\mathbf{o}}(\phi)y)$ into the region $\mathbf{u}(\mathbf{b}_1^{\mathbf{o}}(\phi)z)$;
- $\mathbf{u}(\mathbf{b}_1^m(\phi)g) \equiv$ resulting next the probe, assembled by the species $\mathbf{u}(\mathbf{b})$ and implementing the geometric transformation $\mathbf{u}(\phi)$, applies on the action $\mathbf{u}(g)$ mapping the region $\mathbf{u}(y)$ into the region $\mathbf{u}(z)$;

- $u(T_3(M)(b_1^o(\phi)y)A) \equiv$ resulting next the observable $u(A)$ interacts with the device placed in the region $u(b_1^o(\phi)y)$, assembled by the connector adding the charge T and transporting into $s(b(M))$;
- $s(a(N)) \equiv$ dynamical pattern associated with the context $u(N)$ and implemented by the species $u(a)$;
- $s(b(M)) \equiv$ dynamical pattern associated with the context $u(M)$ and implemented by the species $u(b)$.

DEFINITION 1.4.36. τ is the conjugate over the second place of variability of a map of maps.

PROPOSITION 1.4.37 (Equiformity principle equivariant form). Under the hypothesis of Prp. 1.4.34 we have

$$(a_3^+(\phi)(T_1^o(N)y) \circ (T_1^m(N) \upharpoonright R_N^b(y, z))^\tau) \dagger^{a, N, T_1^o(N)y, T_1^o(N)z, T_3^+(N)(z)\psi} = \\ (T_3^+(M)(b_1^o(\phi)y) \circ (b_1^m(\phi) \upharpoonright R_N^b(y, z))^\tau) \dagger^{b, M, b_1^o(\phi)y, b_1^o(\phi)z, b_3^+(\phi)(z)\psi}.$$

Let us introduce some concept to provide in Prp. 1.4.39 the species-context meaning of the equiformity principle in terms of trajectories.

DEFINITION 1.4.38. Let \mathcal{D} be a category, $a, b \in \mathbf{Fct}(\mathcal{D}, \mathcal{U}h\mathbf{dv})$, $\mathfrak{E}^a = (\mathfrak{S}^a, R^a) \in \mathbf{Exp}(a)$, $M, N \in \mathcal{D}$ and $\phi \in \mathbf{Mor}_{\mathcal{D}}(M, N)$. Let $(T, a, \phi, \mathfrak{E}^a)$ -transformation be any transformation on $\mathfrak{f}^{a, N}$ composition of the conjugate of a transformation valued in $G_{a(N)} \times G_{a(N)}$ and implemented by $T_1^o(N) \times T_1^o(N)$, composed the second conjugate of a transformation valued in $\mathfrak{S}_N^a(y) \times \mathcal{A}_{a(N)}(x)_{ob}$ for some $x, y \in G_{a(N)}$ and implemented by $T_3^+(N)(\cdot) \times a_3(\phi)(T_1^o(N)(\cdot))$, composed the third conjugate of a transformation valued in $R_N^a(u, v)$ for some $u, v \in G_{a(N)}$ and implemented by $T_1^m(N)$. Let $(b, \phi, T, \mathfrak{E}^b)$ -transformation be any transformation on $\mathfrak{f}^{b, M}$ composition of the conjugate of a transformation valued in $G_{b(M)} \times G_{b(M)}$ and implemented by $b_1^o(\phi)(\cdot) \times b_1^o(\phi)(\cdot)$, composed the second conjugate of a transformation valued in $\mathfrak{S}_M^b(y) \times \mathcal{A}_{b(M)}(x)_{ob}$ for some $x, y \in G_{b(M)}$ and implemented by $b_3^+(\phi)(\cdot) \times T_3(M)(b_1^o(\phi)(\cdot))$, composed the third conjugate of a transformation valued in $R_M^b(u, v)$ for some $u, v \in G_{b(M)}$ and implemented by $b_1^m(\phi)$.

PROPOSITION 1.4.39 (Species-context meaning of the equiformity principle). Let \mathcal{D} be a category, $a, b \in \mathbf{Fct}(\mathcal{D}, \mathcal{U}h\mathbf{dv})$, $\mathfrak{E}^a \in \mathbf{Exp}(a)$, $\mathfrak{E}^b \in \mathbf{Exp}(b)$, and T be a link from \mathfrak{E}^b to \mathfrak{E}^a . Thus for all $M, N \in \mathcal{D}$ and $\phi \in \mathbf{Mor}_{\mathcal{D}}(M, N)$ the action on the trajectory of **the species $u(a)$ in the context $u(N)$** of a $(T, a, \phi, \mathfrak{E}^a)$ -transformation equals the action on the trajectory of **the species $u(b)$ in the context $u(M)$** of a $(b, \phi, T, \mathfrak{E}^b)$ -transformation.

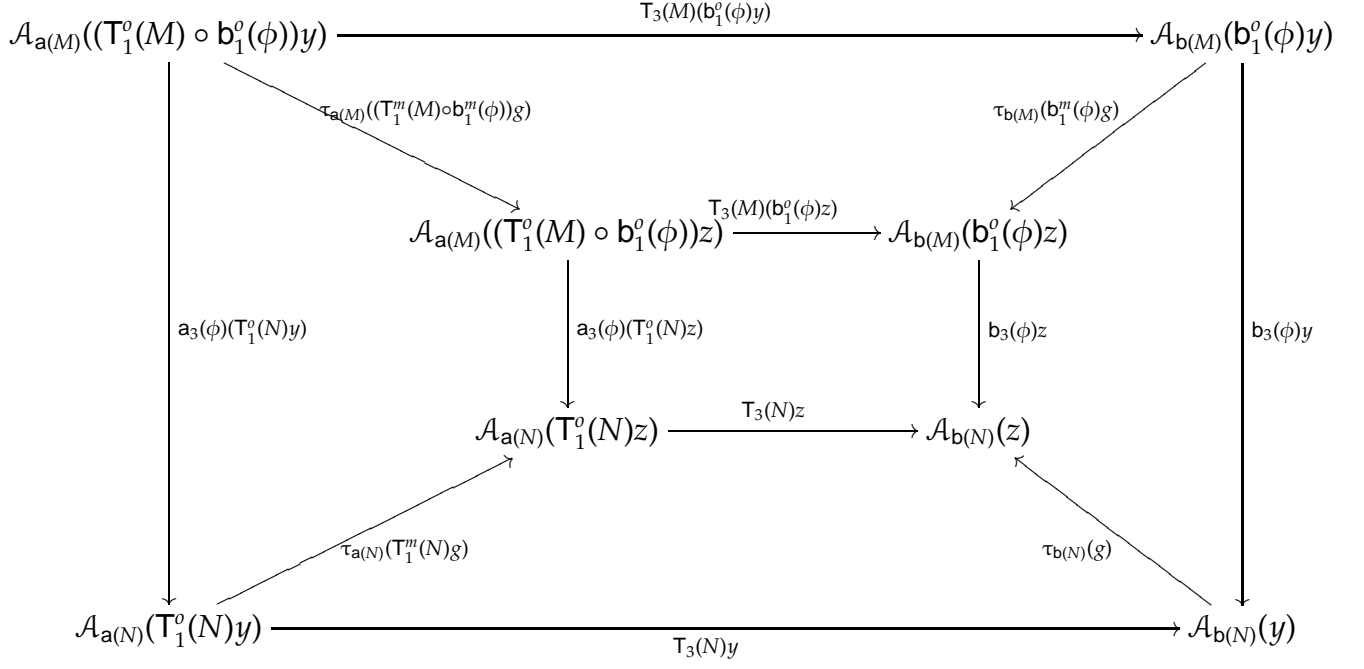
PROPOSITION 1.4.40 (Equiformity principle for propensities). In addition to the hypothesis of Prp. 1.4.34 let $\psi(1) \neq 0$ and $e \in \mathbf{Ef}(\mathcal{A}_{a(M)}((T_1^o(M) \circ b_1^o(\phi))y))$ thus

$$\delta_{\substack{a, N, T_1^o(N)y, T_1^o(N)z \\ (T_3^+(N)(z), \psi, a_3(\phi)(T_1^o(N)y)e)}} (T_1^m(N)g) = \delta_{\substack{b, M, b_1^o(\phi)y, b_1^o(\phi)z \\ (b_3^+(\phi)(z), \psi, T_3(M)(b_1^o(\phi)y)e)}} (b_1^m(\phi)g).$$

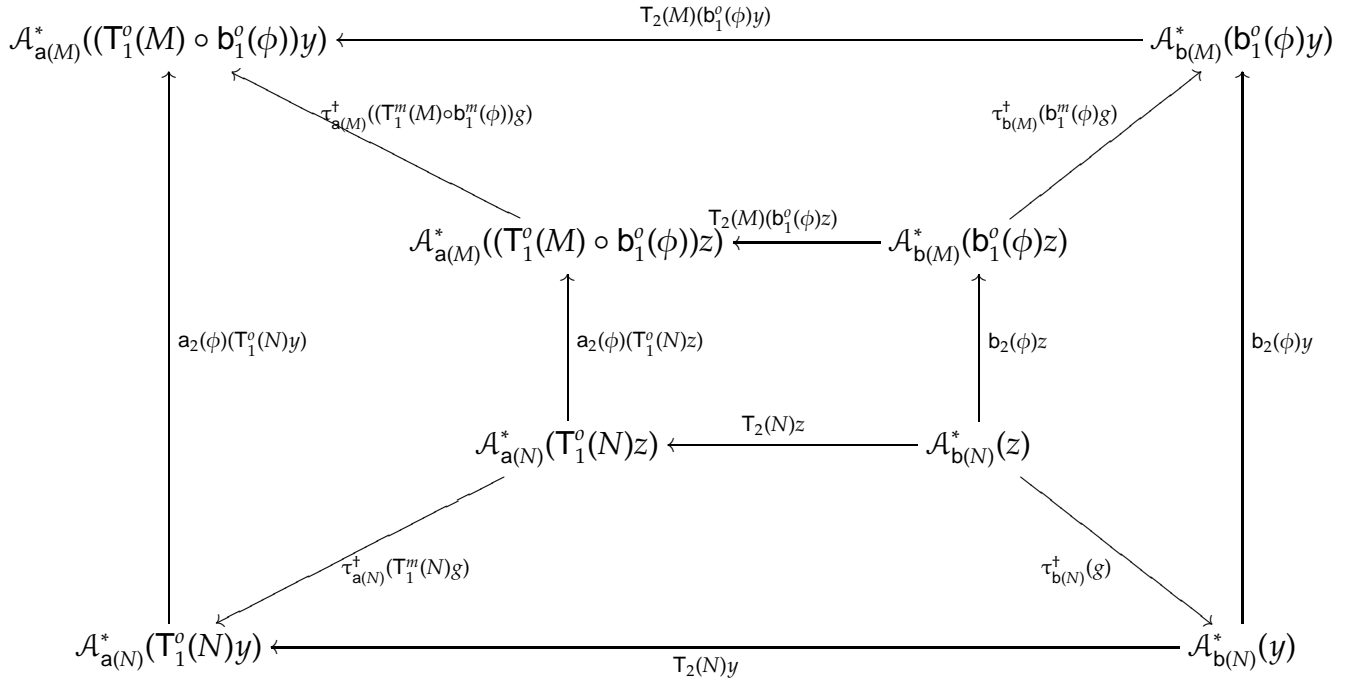
The next is a preparatory result to Thm. 1.4.46

LEMMA 1.4.41. Let \mathcal{D} be a category, $a, b \in \mathbf{Fct}(\mathcal{D}, \mathcal{U}h\mathbf{dv})$, and $T \in \mathbf{Mor}_{\mathbf{Fct}(\mathcal{D}, \mathcal{U}h\mathbf{dv})}(a, b)$ then for all $M, N \in \mathcal{D}$, $\phi \in \mathbf{Mor}_{\mathcal{D}}(M, N)$, $y, z \in G_{b(N)}$ and $g \in G_{b(N)}(y, z)$ the following is a commutative

diagram in ptsa



and the following is a commutative diagram in ptls



PROOF. Since T is a natural transformation from a to b , the following is a commutative diagram in $\mathcal{C}h\mathfrak{d}v$ for all $M, N \in \mathfrak{D}$ and $\phi \in Mor_{\mathfrak{D}}(M, N)$

$$(20) \quad \begin{array}{ccc} a(M) & \xrightarrow{T(M)} & b(M) \\ \downarrow a(\phi) & & \downarrow b(\phi) \\ a(N) & \xrightarrow{T(N)} & b(N). \end{array}$$

Next let us call square c the central subsquare in the first diagram of the statement, and square u , d , l and r the up, down, left and right subsquare of the first diagram in the statement respectively; moreover let \star denote the commutativity of the diagram in Rmk.

1.4.3. The squares u and d are commutative since \star and the arrows $a(M) \xrightarrow{T(M)} b(M)$ and $a(N) \xrightarrow{T(N)} b(N)$. The square l is commutative since \star , the arrow $a(M) \xrightarrow{a(\phi)} a(N)$ and the 1th component of the commutativity of the diagram (20), i.e.

$$(21) \quad T_1(M) \circ b_1(\phi) = a_1(\phi) \circ T_1(N).$$

The square r is commutative since \star and the arrow $b(M) \xrightarrow{b(\phi)} b(N)$, finally the square c is commutative since the 3th component of the commutativity of the diagram in (20), i.e. $b_3(\phi) \circ (T_3(M) * 1_{b_1(\phi)}) = T_3(N) \circ (a_3(\phi) * 1_{T_1(N)})$. By denoting with \natural the commutativity of the diagram in Rmk. 1.4.9 we obtain the commutativity of the second diagram in the statement just by following the same line of reasoning used for the first one, by replacing \star by \natural and by taking the 2th component of the commutativity of the diagram in (20) instead of the 3th one. \square

PROPOSITION 1.4.42. *Under the hypothesis of Lemma 1.4.41 we have that*

(1) *if ϕ is an isomorphism, then for all $h \in G_{b(M)}(b_1^o(\phi^{-1})y, b_1^o(\phi^{-1})z)$*

$$\tau_{b(N)}(b_1^m(\phi^{-1})h) \circ (b_3(\phi)y) = (b_3(\phi)z) \circ \tau_{b(M)}(h);$$

(2) *if $G_{b(N)}(y, z) = G_{a(N)}(T_1^o(N)y, T_1^o(N)z)$ and $T_1^m(N) \upharpoonright G_{b(N)}(y, z) = 1_{G_{b(N)}(y, z)}$, then*

$$\tau_{b(N)}(g) \circ (T_3(N)y) = (T_3(N)z) \circ \tau_{a(N)}(g).$$

PROOF. $b(\phi^{-1}) = b(\phi)^{-1}$ in the category $\mathcal{C}h\mathfrak{d}v$ since b is a functor, in particular $b_1^o(\phi^{-1}) = b_1^o(\phi)^{-1}$ and $b_1^m(\phi^{-1}) = b_1^m(\phi)^{-1}$. Then sts. (1) and (2) follow since the right and the bottom squares of the first commutative diagram in Lemma 1.4.41 respectively. \square

Let us comment the above result. In [13] given a theory namely a functor say \mathcal{B} from the category of globally hyperbolic spacetimes Loc to $Phys$ one obtains that:

- (1) The relative Cauchy evolution $rce_M^{\mathcal{B}}$ for any object M of \mathbf{Loc} is a geometric object: *It depends uniquely by the morphism map of \mathcal{B} , being a composition of four its factorizations see [13, p.16].*
- (2) $rce^{\mathcal{B}}$ is equivariant under action of the morphisms of \mathbf{Loc} implemented by \mathcal{B} [13, Prp. 3.7(eq. 3.6)].
- (3) Given a theory \mathcal{A} and a suitable natural transformation ζ from \mathcal{A} to \mathcal{B} , $rce^{\mathcal{A}}$ and $rce^{\mathcal{B}}$ are ζ -related [13, Prp. 3.8].

Instead in our approach the concept of dynamics is structured in a wholly general sense in the object set of \mathfrak{dp} , hence in that of the target category \mathfrak{Chdv} , rather than constructed via the morphism set of the source category. Thus for a fixed but general functor \mathbf{b} from a category \mathfrak{D} to the category \mathfrak{Chdv} one obtains that:

- (1) The dynamics $\tau_{\mathbf{b}(M)}$ for any object M of \mathfrak{D} is *not* a geometric object. $\tau_{\mathbf{b}(M)}$ does **not** present any other relationship with \mathbf{b}_3 other than the symmetries appearing in the first diagram in Lemma 1.4.41. In particular $\tau_{\mathbf{b}(M)}$ does **not** factorizes through \mathbf{b}_3 .
- (2) $\tau_{\mathbf{b}(\cdot)}$ is equivariant under action of the morphisms of \mathfrak{D} implemented by \mathbf{b}_3 as shown in the right square of the first diagram in Lemma 1.4.41. As a particular result Prp. 1.4.42(1) follows which can be considered a generalization of [13, Prp. 3.7 (eq. 3.6)].
- (3) Given a functor \mathbf{a} from \mathfrak{D} to \mathfrak{Chdv} and a natural transformation \mathbf{T} from \mathbf{a} to \mathbf{b} , $\tau_{\mathbf{a}(\cdot)}$ and $\tau_{\mathbf{b}(\cdot)}$ are \mathbf{T} -related as shown in the bottom square of the first diagram in Lemma 1.4.41. As a particular result Prp. 1.4.42(2) follows which can be considered a generalization of [13, Prp. 3.8].
- (4) But the fundamental point is that Prp. 1.4.42(1) and Prp. 1.4.42(2) are not uncorrelated symmetries, rather they emerge as very specific outcomes of the unique general **equiformity principle of \mathbf{T}** Thm. 1.4.46(5). This principle follows by the *entire first diagram in Lemma 1.4.41* which has not correspondence in [13].

In the next definition we introduce the component required in order to state Thm. 1.4.46.

DEFINITION 1.4.43. *Let \mathfrak{D} be a category, $\mathbf{a}, \mathbf{b} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, $\mathfrak{E} = (\mathcal{S}, \mathcal{R}) \in \mathbf{Exp}(\mathbf{b})$ and $\mathbf{T} \in \mathbf{Mor}_{\mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})}(\mathbf{a}, \mathbf{b})$. Let*

$$\Gamma(\mathfrak{E}, \mathbf{T}) := \{s \in \prod_{M \in \mathfrak{D}} \mathbf{Mor}_{\mathbf{set}}(\mathbf{T}_1^o(M)(\mathbf{Obj}(\mathbf{R}_M)), \mathbf{Obj}(\mathbf{R}_M)) \mid (1, 2)\},$$

where

- (1) $s_M \circ \mathbf{T}_1^o(M) = \mathbf{1}_{c(s_M)}$ for all $M \in \mathfrak{D}$;
- (2) $\mathbf{b}_1^o(\phi) \circ s_N = s_M \circ \mathbf{a}_1^o(\phi)$, for all $M, N \in \mathfrak{D}$ and $\phi \in \mathbf{Mor}_{\mathfrak{D}}(M, N)$.

Next let $s \in \Gamma(\mathfrak{E}, \mathbf{T})$ define $\mathbf{T}[\mathbf{R}, s]$ to be the map on $\mathbf{Obj}(\mathfrak{D})$ mapping any $M \in \mathfrak{D}$ to the unique subcategory $\mathbf{T}[\mathbf{R}, s]_M$ of $\mathbf{G}_M^{\mathbf{a}}$ such that

$$\begin{aligned} \mathbf{Obj}(\mathbf{T}[\mathbf{R}, s]_M) &= \mathbf{T}_1^o(M)(\mathbf{Obj}(\mathbf{R}_M)), \\ \mathbf{Mor}_{\mathbf{T}[\mathbf{R}, s]_M}(u, t) &= \mathbf{T}_1^m(M)(\mathbf{Mor}_{\mathbf{R}_M}(s_M(u), s_M(t))), \\ &\forall u, t \in \mathbf{Obj}(\mathbf{T}[\mathbf{R}, s]_M). \end{aligned}$$

Moreover define

$$\mathsf{T}[\mathcal{S}, s] \in \prod_{M \in \mathcal{D}} \prod_{t \in \mathsf{T}[\mathcal{R}, s]_M} \mathcal{P}(\mathfrak{P}_M^{\mathbf{a}}(t)),$$

such that for all $M \in \mathcal{D}$ and $t \in \mathsf{T}[\mathcal{R}, s]_M$

$$\mathsf{T}[\mathcal{S}, s]_M(t) := \mathsf{T}_3^+(M)(s_M(t))(s_M(s_M(t))).$$

Finally set $\mathsf{T}[\mathfrak{E}, s] := (\mathsf{T}[\mathcal{S}, s], \mathsf{T}[\mathcal{R}, s])$.

REMARK 1.4.44. Assume the notation of Def. 1.4.43, since (21) and $\mathfrak{E} \in \text{Exp}(\mathbf{b})$ we deduce that $\mathbf{a}_1^o(\phi)\mathsf{T}_1^o(N)(\text{Obj}(\mathcal{R}_N)) \subseteq \mathsf{T}_1^o(M)(\text{Obj}(\mathcal{R}_M))$ rendering Def. 1.4.43(2) consistent.

REMARK 1.4.45. Assume the notation of Def. 1.4.43, and that $\mathsf{T}_1^o(M)$ is injective for all $M \in \mathcal{D}$, then since (21) we have that $M \mapsto \mathsf{T}_1^o(M)^{-1} \upharpoonright \mathsf{T}_1^o(M)(\text{Obj}(\mathcal{R}_M)) \in \Gamma(\mathfrak{E}, \mathsf{T})$, called the projection associated with \mathfrak{E} and T .

The next is the first main result of this paper. For any connector T , namely a natural transformation between two species contextualized on the same category, any experimental setting \mathcal{Q} of the target species of T and any $s \in \Gamma(\mathcal{Q}, \mathsf{T})$ we prove that the above defined $\mathsf{T}[\mathcal{Q}, s]$ is an experimental setting of the source species of T and that T is a link from \mathcal{Q} to $\mathsf{T}[\mathcal{Q}, s]$, consequently with T remains associated a collection of equiformity principles.

THEOREM 1.4.46 (Experimental settings generated and links assembled via natural transformations). *Let \mathcal{D} be a category, $\mathbf{a}, \mathbf{b} \in \text{Fct}(\mathcal{D}, \mathfrak{Ch}\mathfrak{dv})$, $\mathfrak{E} \in \text{Exp}(\mathbf{b})$ and $\mathsf{T} \in \text{Mor}_{\text{Fct}(\mathcal{D}, \mathfrak{Ch}\mathfrak{dv})}(\mathbf{a}, \mathbf{b})$ such that $\Gamma(\mathfrak{E}, \mathsf{T}) \neq \emptyset^{11}$. Thus for all $s \in \Gamma(\mathfrak{E}, \mathsf{T})$*

- (1) $\mathsf{T}[\mathfrak{E}, s] \in \text{Exp}(\mathbf{a})$;
- (2) T is a link from \mathfrak{E} to $\mathsf{T}[\mathfrak{E}, s]$;
- (3) if either $\mathsf{T}_3^+ = \mathsf{T}_2$ or \mathbf{a} factorizes through Ψ then $\mathsf{T}[\mathfrak{E}, s]$ is complete;
- (4) if $\mathfrak{E} \in \text{Exp}^*(\mathbf{b})$, then $\mathsf{T}[\mathfrak{E}, s] \in \text{Exp}^*(\mathbf{a})$;
- (5) the statements of Prp. 1.4.34, Prp. 1.4.37 and Prp. 1.4.40 hold true by replacing $\mathfrak{E}^{\mathbf{a}}$ by $\mathsf{T}[\mathfrak{E}, s]$ and $\mathfrak{E}^{\mathbf{b}}$ by \mathfrak{E} .

PROOF. In what follows whenever we mention to show items of Def. 1.4.24 and Def. 1.4.27 we mean to prove the corresponding statements for $\mathsf{T}[\mathfrak{E}, s]$ and T respectively. Def. 1.4.24(3a) follows since (21) and Def. 1.4.43(2). By the external square in the first diagram in Lemma 1.4.41 and since Lemma 1.3.1(2) we deduce for all $M, N \in \mathcal{D}$, $\phi \in \text{Mor}_{\mathcal{D}}(M, N)$ and $y \in G_N^{\mathbf{b}}$ that

$$\mathbf{a}_3^+(\phi)(\mathsf{T}_1^o(N)y) \circ \mathsf{T}_3^+(N)(y) = \mathsf{T}_3^+(M)(\mathbf{b}_1^o(\phi)y) \circ \mathbf{b}_3^+(\phi)(y).$$

Hence by letting $\mathfrak{E} = (\mathcal{S}, \mathcal{R})$ and using Def. 1.4.43(1) we have for all $t \in \mathsf{T}[\mathcal{R}, s]_N$

$$\mathbf{a}_3^+(\phi)(t) \circ \mathsf{T}_3^+(N)(s_N(t)) = \mathsf{T}_3^+(M)(\mathbf{b}_1^o(\phi)s_N(t)) \circ \mathbf{b}_3^+(\phi)(s_N(t)),$$

¹¹for instance whenever $\mathsf{T}_1^o(M)$ is injective for all $M \in \mathcal{D}$ see Rmk. 1.4.45.

therefore

$$\begin{aligned}
a_3^+(\phi)(t)T[\mathcal{S}, s]_N(t) &= (a_3^+(\phi)(t) \circ T_3^+(N)(s_N(t)))\mathcal{S}_N(s_N(t)) \\
&= (T_3^+(M)(b_1^o(\phi)_{s_N(t)}) \circ b_3^+(\phi)(s_N(t)))\mathcal{S}_N(s_N(t)) \\
&\subseteq T_3^+(M)(b_1^o(\phi)_{s_N(t)})\mathcal{S}_M(b_1^o(\phi)_{s_N(t)}) \\
&= T_3^+(M)(s_M(a_1^o(\phi)t))\mathcal{S}_M(s_M(a_1^o(\phi)t)) = T[\mathcal{S}, s]_M(a_1^o(\phi)t),
\end{aligned}$$

which proves Def. 1.4.24(3b). Since Rmk. 1.4.3 and Lemma 1.3.1(2) we have for any $y, z \in G_M^b$ and $g \in G_M^b(y, z)$ that

$$\tau_{a(M)}^+(T_1^m(M)g) \circ T_3^+(M)(z) = T_3^+(M)(y) \circ \tau_{b(M)}^+(g).$$

Let $u, t \in T[\mathcal{R}, s]_M$ and $h \in T[\mathcal{R}, s]_M(u, t)$, hence there exists $g \in R_M(s_M(u), s_M(t))$ such that $h = T_1^m(M)g$, therefore

$$\begin{aligned}
\tau_{a(M)}^+(h)T[\mathcal{S}, s]_M(t) &= (\tau_{a(M)}^+(T_1^m(M)g) \circ T_3^+(M)(s_M(t)))\mathcal{S}_M(s_M(t)) \\
&= (T_3^+(M)(s_M(u)) \circ \tau_{b(M)}^+(g))\mathcal{S}_M(s_M(t)) \\
&\subseteq T_3^+(M)(s_M(u))\mathcal{S}_M(s_M(u)) = T[\mathcal{S}, s]_M(u),
\end{aligned}$$

so Def. 1.4.24(4) and st.(1) follows. Next Def. 1.4.27 follows since construction of $T[\mathfrak{E}]$ and Lemma 1.4.41, and st.(2) follows. If \mathbf{a} factorizes through the functor Ψ then $\mathbf{a}_2 = \mathbf{a}_3^+$ and the second option in st.(3) follows, while the first one can be proven in the same way of st.(1) by using instead the second diagram in Lemma 1.4.41. For all $M \in \mathfrak{D}$, $t \in T[\mathcal{R}, s]_M$ and $a \in \mathcal{A}_{a(M)}(t)$ we have

$$\gamma_a^+ \circ T_3^+(M)(s_M(t)) = T_3^+(M)(s_M(t)) \circ \gamma_{T_3^+(M)(s_M(t))a'}^+$$

hence st.(4) follows since st.(1). St.(5) follows since st.(1) and Prps. 1.4.34, Prp. 1.4.37 and 1.4.40. \square

If we call charge from \mathbf{b} to \mathbf{a} any map from a possibly subset of $Exp(\mathbf{b})$ to $Exp(\mathbf{a})$, we obtain that Thm. 1.4.46(1) establishes that $T[\cdot, s]$ is a charge from \mathbf{b} to \mathbf{a} . In case of species of dynamical systems we shall see in [31] that the subset extends to the whole $Exp(\mathbf{b})$. We can avoid the use of the map Γ in the following case.

THEOREM 1.4.47. *Let \mathfrak{D} be a category, $\mathbf{a}, \mathbf{b} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chddv})$, $\mathfrak{E} \in Exp(\mathbf{b})$ and $T \in Mor_{\mathbf{Fct}(\mathfrak{D}, \mathfrak{Chddv})}(\mathbf{a}, \mathbf{b})$. Thus*

- (1) T is a link from \mathfrak{E} to (\mathfrak{P}^a, G^a) ;
- (2) the statements of Prp. 1.4.34, Prp. 1.4.37 and Prp. 1.4.40. hold true by replacing \mathfrak{E}^a by (\mathfrak{P}^a, G^a) and \mathfrak{E}^b by \mathfrak{E} .

PROOF. St.(1) follows since Lemma 1.4.41, st.(2) follows since st.(1), Prp. 1.4.34, Prp. 1.4.37 and Prp. 1.4.40. \square

Thm. 1.4.46(1) establishes that $T[\cdot, s]$ is a charge from the target to the source of T , therefore it is natural to expect that under suitable conditions, the vertical composition

of connectors is contravariantly represented as composition of charges. This is proven in the following second main result.

COROLLARY 1.4.48 (Charge composition of connectors). *Let \mathfrak{D} be a category, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, $\mathfrak{Q} \in \mathbf{Exp}(\mathbf{c})$, $\mathbf{T} \in \mathbf{Mor}_{\mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})}(\mathbf{b}, \mathbf{c})$, $\mathbf{S} \in \mathbf{Mor}_{\mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})}(\mathbf{a}, \mathbf{b})$ and $r \in \Gamma(\mathfrak{Q}, \mathbf{T})$. Thus*

$$r \circ \Gamma(\mathbf{T}[\mathfrak{Q}, r], \mathbf{S}) \subseteq \Gamma(\mathfrak{Q}, \mathbf{T} \circ \mathbf{S}),$$

where the composition is defined pointwise. Moreover if $\mathbf{T}_1^o(M)$ is injective for all $M \in \mathfrak{D}$, and r is the projection associated with \mathfrak{Q} and \mathbf{T} , Rmk. 1.4.45, then for any $q \in \Gamma(\mathbf{T}[\mathfrak{Q}, r], \mathbf{S})$ we obtain

$$(\mathbf{T} \circ \mathbf{S})[\mathfrak{Q}, r \circ q] = \mathbf{S}[\mathbf{T}[\mathfrak{Q}, r], q].$$

PROOF. The inclusion in the statement is well-set since Thm. 1.4.46, then it follows by a simple application of the definitions involved. The equality in the statement is well-set since the inclusion. The part concerning the object sets in the equality follows by direct application of the involved definitions, let us prove the part concerning the statistical ensemble spaces. Let $\mathfrak{Q} = (\mathfrak{S}, \mathbf{R})$, by using Rmk. 1.4.26 we obtain for all $M \in \mathfrak{D}$

$$\begin{aligned} (\mathbf{T} \circ \mathbf{S})[\mathfrak{S}, r \circ q]_M(t) &= (\mathbf{T} \circ \mathbf{S})_3^+(M)((rq)_M(t)) \mathfrak{S}_M((rq)_M(t)) \\ &= (\mathbf{T}_3(M)((rq)_M(t)) \circ \mathbf{S}_3(M)(\mathbf{T}_1^o(rq)_M(t)))^\dagger \mathfrak{S}_M((rq)_M(t)) \\ &= (\mathbf{S}_3^+(M)(q_M(t)) \circ \mathbf{T}_3^+(M)((rq)_M(t))) \mathfrak{S}_M((rq)_M(t)). \end{aligned}$$

Next

$$\begin{aligned} \mathbf{S}[\mathbf{T}[\mathfrak{S}, r], q]_M(t) &= \mathbf{S}_3^+(M)(q_M(t)) \mathbf{T}[\mathfrak{S}, r]_M(q_M(t)) \\ &= (\mathbf{S}_3^+(M)(q_M(t)) \circ \mathbf{T}_3^+(M)((rq)_M(t))) \mathfrak{S}_M((rq)_M(t)). \end{aligned}$$

□

Next we prepare for another important consequence of Thm. 1.4.46 stated in Cor. 1.4.53 until which we let $\mathfrak{G}, \mathfrak{D}$ be categories, $\mathbf{x}, \mathbf{y} \in \mathbf{Fct}(\mathfrak{G}, \mathfrak{D})$, $\mathbf{a}, \mathbf{b} \in \mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})$, $\mathbf{L} \in \mathbf{Mor}_{\mathbf{Fct}(\mathfrak{G}, \mathfrak{D})}(\mathbf{x}, \mathbf{y})$ and $\mathbf{T} \in \mathbf{Mor}_{\mathbf{Fct}(\mathfrak{D}, \mathfrak{Chdv})}(\mathbf{a}, \mathbf{b})$.

DEFINITION 1.4.49. For any $\mathfrak{Q} = (\mathfrak{S}, \mathbf{R}) \in \mathbf{Exp}(\mathbf{b})$ and $s \in \Gamma(\mathfrak{Q}, \mathbf{T})$ define

$$\Xi(\mathbf{T}, \mathbf{L}, s) := \{r \in \prod_{G \in \mathfrak{G}} \mathbf{Mor}_{\mathbf{set}}(\mathbf{a}_1^o(\mathbf{L}(G)) \circ \mathbf{T}_1^o(\mathbf{y}(G)) \mathbf{Obj}(\mathbf{R}_{\mathbf{y}(G)}), \mathbf{Obj}(\mathbf{R}_{\mathbf{y}(G)})) \mid (1, 2) \text{ hold true}\},$$

where

- (1) $r_G \circ \mathbf{a}_1^o(\mathbf{L}(G)) \circ (\mathbf{T}_1^o(\mathbf{y}(G))(\mathbf{Obj}(\mathbf{R}_{\mathbf{y}(G)}))) \hookrightarrow \mathbf{G}_{\mathbf{y}(G)}^{\mathbf{a}} = s_{\mathbf{y}(G)}$, for all $G \in \mathfrak{G}$;
- (2) $\mathbf{b}_1^o(\mathbf{L}(G)) \circ r_G = s_{\mathbf{x}(G)}$, for all $G \in \mathfrak{G}$.

Moreover set $\mathbf{y}[\mathfrak{Q}] := (\mathfrak{S} \circ \mathbf{y}^o, \mathbf{R} \circ \mathbf{y}^o)$.

REMARK 1.4.50. For all $G \in \mathfrak{G}$ by applying the general definition of Godement product (horizontal composition) we have $(\mathbf{T} * \mathbf{L})(G) = \mathbf{T}(\mathbf{y}(G)) \circ \mathbf{a}(\mathbf{L}(G)) = \mathbf{b}(\mathbf{L}(G)) \circ \mathbf{T}(\mathbf{x}(G))$. Hence by the definition of morphism composition in \mathfrak{Chdv} we obtain $(\mathbf{T} * \mathbf{L})_1(G) = \mathbf{a}_1(\mathbf{L}(G)) \circ \mathbf{T}_1(\mathbf{y}(G)) = \mathbf{T}_1(\mathbf{x}(G)) \circ \mathbf{b}_1(\mathbf{L}(G))$, and $(\mathbf{T} * \mathbf{L})_3(G) = \mathbf{T}_3(\mathbf{y}(G)) \circ (\mathbf{a}_3(\mathbf{L}(G)) * \mathbf{1}_{\mathbf{T}_1(\mathbf{y}(G))})$, where we recall (2).

LEMMA 1.4.51. *For any $\mathfrak{Q} \in \text{Exp}(\mathbf{b})$ and $s \in \Gamma(\mathfrak{Q}, \mathbf{T})$ we have that*

- (1) $y[\mathfrak{Q}] \in \text{Exp}(\mathbf{b} \circ y)$;
- (2) $\Xi(\mathbf{T}, \mathbf{L}, s) \subset \Gamma(y[\mathfrak{Q}], \mathbf{T} * \mathbf{L})$.

PROOF. St. (1) follows by the functoriality of y and the definition of experimental settings, st. (2) follows since Rmk. 1.4.50 and Def. 1.4.49(1). \square

Often we call $y[\mathfrak{Q}]$ the pullback of \mathfrak{Q} through y . Next we define an order relation in the collection of the experimental settings.

DEFINITION 1.4.52. *Let \leq_a or simply \leq be the relation on $\text{Exp}(\mathbf{a})$ defined such that for all $(\mathfrak{S}, \mathbf{R}), (\mathfrak{S}', \mathbf{R}') \in \text{Exp}(\mathbf{a})$ we have $(\mathfrak{S}, \mathbf{R}) \leq (\mathfrak{S}', \mathbf{R}')$ iff \mathbf{R}_M is a subcategory of \mathbf{R}'_M and $\mathfrak{S}_M(t) \subseteq \mathfrak{S}'_M(t)$ for all $t \in \mathbf{R}_M$ and $M \in \mathfrak{D}$.*

In addition to the contravariant representation of the vertical composition of connectors in terms of composition of charges, if $T_1^o(M)$ is injective for all $M \in \mathfrak{D}$, we have also a representation of the horizontal composition of a connector \mathbf{T} with a natural transformation \mathbf{L} in terms of what we call charge transfer. Said x and y the source and target of \mathbf{L} respectively, the charge transfer roughly asserts that the charge $(\mathbf{T} * \mathbf{L})[\cdot, r]$ for $r \in \Xi(\mathbf{T}, \mathbf{L}, s)$ maps the pullback through y of any experimental setting \mathfrak{Q} of the target species of \mathbf{T} into an experimental setting which is included in the pullback through x of the experimental setting obtained by mapping \mathfrak{Q} through the charge $\mathbf{T}[\cdot, s]$, where s is the projection associated with \mathfrak{Q} and \mathbf{T} . The following is the third main result of this paper.

COROLLARY 1.4.53 (Charge transfer). *If $T_1^o(M)$ is injective for all $M \in \mathfrak{D}$, then for any $\mathfrak{Q} \in \text{Exp}(\mathbf{b})$ said s the projection associated with \mathfrak{Q} and \mathbf{T} Rmk. 1.4.45, we obtain for all $r \in \Xi(\mathbf{T}, \mathbf{L}, s)$ that*

$$(\mathbf{T} * \mathbf{L})[y[\mathfrak{Q}], r] \leq x[\mathbf{T}[\mathfrak{Q}, s]].$$

PROOF. Let K denote $(\mathbf{T} * \mathbf{L})[\mathbf{R} \circ y^o, r]$, \mathfrak{X} denote $(\mathbf{T} * \mathbf{L})[\mathfrak{S} \circ y^o, r]$, and let $\mathfrak{Q} = (\mathfrak{S}, \mathbf{R})$. Thus

$$(22) \quad \begin{aligned} (\mathbf{T} * \mathbf{L})[y[\mathfrak{Q}], r] &= (\mathfrak{X}, K), \\ x[\mathbf{T}[\mathfrak{Q}, s]] &= (\mathbf{T}[\mathfrak{S}, s] \circ x^o, \mathbf{T}[\mathbf{R}, s] \circ x^o). \end{aligned}$$

Moreover for any $G \in \mathfrak{G}$

$$\begin{aligned} \text{Obj}(K_G) &= (\mathbf{T} * \mathbf{L})_1^o(G)(\text{Obj}(\mathbf{R}_{y(G)})) \\ &= (T_1^o(x(G)) \circ b_1^o(L(G)))\text{Obj}(\mathbf{R}_{y(G)}) \\ &\subseteq T_1^o(x(G))\text{Obj}(\mathbf{R}_{x(G)}) \\ &= \text{Obj}((\mathbf{T}[\mathbf{R}, s] \circ x^o)(G)), \end{aligned}$$

where in the second equality we use Rmk. 1.4.50, the inclusion arises by $\mathfrak{Q} \in \text{Exp}(\mathbf{b})$ and $L(G) \in \text{Mor}_{\mathfrak{D}}(\mathbf{x}(G), \mathbf{y}(G))$. Let $u, t \in \text{Obj}(\mathbf{K}_G)$ thus by Rmk. 1.4.50

$$\begin{aligned} \text{Mor}_{\mathbf{K}_G}(u, t) &= (\mathbf{T} * \mathbf{L})_1^m(G) \text{Mor}_{\mathbf{R}_{\mathbf{y}(G)}}(r_G u, r_G t) \\ &= (\mathbf{T}_1^m(\mathbf{x}(G)) \circ \mathbf{b}_1^m(\mathbf{L}(G))) \text{Mor}_{\mathbf{R}_{\mathbf{y}(G)}}(r_G u, r_G t) \\ &\subseteq \mathbf{T}_1^m(\mathbf{x}(G)) \text{Mor}_{\mathbf{R}_{\mathbf{x}(G)}}((\mathbf{b}_1^o(\mathbf{L}(G)) \circ r_G)u, (\mathbf{b}_1^o(\mathbf{L}(G)) \circ r_G)t) \\ &= \mathbf{T}_1^m(\mathbf{x}(G)) \text{Mor}_{\mathbf{R}_{\mathbf{x}(G)}}(s_{\mathbf{x}(G)}u, s_{\mathbf{x}(G)}t) \\ &= \text{Mor}_{(\mathbf{T}[\mathbf{R}, s] \circ \mathbf{x}^o)(G)}(u, t), \end{aligned}$$

where in the inclusion we used $\mathfrak{Q} \in \text{Exp}(\mathbf{b})$ and in the subsequent equality the hypothesis on r . Thus

$$(23) \quad \mathbf{K}_G \text{ is a subcategory of } (\mathbf{T}[\mathbf{R}, s] \circ \mathbf{x}^o)(G).$$

Next

$$\begin{aligned} \mathfrak{X}_G(t) &= (\mathbf{T} * \mathbf{L})_3^+(G)(r_G t) \mathfrak{S}_{\mathbf{y}(G)}(r_G t) \\ &= \left((\mathbf{T} * \mathbf{L})_3(G)(r_G t) \right)^+ \mathfrak{S}_{\mathbf{y}(G)}(r_G t) \\ &= \left(\mathbf{T}_3(\mathbf{y}(G))(r_G t) \circ \mathbf{a}_3(\mathbf{L}(G))(\mathbf{T}_1^o(\mathbf{y}(G))r_G t) \right)^+ \mathfrak{S}_{\mathbf{y}(G)}(r_G t) \\ &= \left(\mathbf{a}_3^+(\mathbf{L}(G))(\mathbf{T}_1^o(\mathbf{y}(G))r_G t) \circ \mathbf{T}_3^+(\mathbf{y}(G))(r_G t) \right) \mathfrak{S}_{\mathbf{y}(G)}(r_G t) \\ &\subseteq \mathbf{a}_3^+(\mathbf{L}(G))(\mathbf{T}_1^o(\mathbf{y}(G))r_G(t)) \mathbf{T}[\mathfrak{S}, s]_{\mathbf{y}(G)}(\mathbf{T}_1^o(\mathbf{y}(G))r_G(t)) \\ &\subseteq \mathbf{T}[\mathfrak{S}, s]_{\mathbf{x}(G)}((\mathbf{a}_1^o(\mathbf{L}(G)) \circ \mathbf{T}_1^o(\mathbf{y}(G)))r_G(t)) \\ &= \mathbf{T}[\mathfrak{S}, s]_{\mathbf{x}(G)}((\mathbf{T}_1^o(\mathbf{x}(G)) \circ \mathbf{b}_1^o(\mathbf{L}(G)))r_G(t)) \\ &= \mathbf{T}[\mathfrak{S}, s]_{\mathbf{x}(G)}((\mathbf{T}_1^o(\mathbf{x}(G)) \circ s_{\mathbf{x}(G)})t) \\ &= \mathbf{T}[\mathfrak{S}, s]_{\mathbf{x}(G)}(t) \\ &= (\mathbf{T}[\mathfrak{S}, s] \circ \mathbf{x}^o)(G)(t), \end{aligned}$$

where we used Rmk. 1.4.50 in the 3th and 5th equalities, the 1th inclusion arises by Thm. 1.4.46(2) and Def. 1.4.27(2), the 2th inclusion by Thm. 1.4.46(1) and Def. 1.4.24(3b), the 6th equality by hypothesis on r and the 7th equality by hypothesis on s . Hence

$$(24) \quad \mathfrak{X}_G(t) \subseteq (\mathbf{T}[\mathfrak{S}, s] \circ \mathbf{x}^o)(G)(t).$$

Finally (22), (23) and (24) yield the statement. \square

1.5. The 2-category 2 – dp

In section 1.5.1 we introduce the general structures utilized in section 1.5.2 in order to construct the 2-category 2 – dp and the fibered category of connectors. Finally in section 1.5.3 we physically interpret the introduced structures.

1.5.1. General definitions. For any category \mathbf{C} we will define the 2-category $2 - \mathbf{C}$ Def. 1.5.7 as a subcategory of \mathbf{Cat} with the same object set and any morphism category $2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ roughly formed by factorizable functors through \mathbf{C} as objects, with morphisms horizontal composition of natural transformations between the corresponding factors **Def. 1.5.1.** In section 1.5.2 we shall apply the construction to the case $\mathbf{C} = \mathbf{dp}$. A standard construction applied to our category $2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ allows to provide $Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g)$ with the structure of a category for $f, g \in 2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ Prp. 1.5.3. These categories are employed to introduce the fibered category of connectors **Def. 1.5.9.**

DEFINITION 1.5.1. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be categories, define $2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ the subcategory of $\mathbf{Fct}(\mathbf{A}, \mathbf{B})$ such that

$$Obj(2 - \mathbf{C}(\mathbf{A}, \mathbf{B})) := \{\Phi \circ \Psi \mid \Psi \in \mathbf{Fct}(\mathbf{A}, \mathbf{C}), \Phi \in \mathbf{Fct}(\mathbf{C}, \mathbf{B})\},$$

where \circ is the standard composition of functors, while for any $f, g \in 2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$,

$$Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g) := \{\beta * \alpha \mid \beta \in Mor_{\mathbf{Fct}(\mathbf{C}, \mathbf{B})}(\Phi_1, \Phi_2), \alpha \in Mor_{\mathbf{Fct}(\mathbf{A}, \mathbf{C})}(\Psi_1, \Psi_2); \\ \Psi_i \in \mathbf{Fct}(\mathbf{A}, \mathbf{C}), \Phi_i \in \mathbf{Fct}(\mathbf{C}, \mathbf{B}), i \in \{1, 2\}; \Phi_1 \circ \Psi_1 = f, \Phi_2 \circ \Psi_2 = g\}.$$

The definition of $Mor_{2-\mathbf{C}}(\mathbf{A}, \mathbf{B})$ is well-done indeed the morphism composition law in $Mor_{2-\mathbf{C}}(\mathbf{A}, \mathbf{B})$ inherited by the one in $\mathbf{Fct}(\mathbf{A}, \mathbf{B})$ is an internal operation by using [3, Prp. 1.3.5]. Since [17, Def. 1.2.5] the set of morphisms of any category can be provided with the structure of category, then in the special case of the category $2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ we have that $Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g)$ is a subcategory of the category $Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}$ that does not provide $2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ the structure of 2-category. However we will prove in Prp. 1.5.5 the existence of a partial product between the morphism sets of these categories. Next [17, Def. 1.2.5] applied to our category $2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ reads as follows

DEFINITION 1.5.2. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be categories, $f, g \in 2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$ and $\delta, \varepsilon, \eta \in Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g)$, define

$$(f, g)\langle \delta, \varepsilon \rangle := \{(u, v) \in Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, f) \times Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(g, g) \mid \varepsilon \circ u = v \circ \delta\},$$

moreover define

$$(\circ) : (f, g)\langle \varepsilon, \eta \rangle \times (f, g)\langle \delta, \varepsilon \rangle \rightarrow (f, g)\langle \delta, \eta \rangle$$

such that $(w, s) \circ (u, v) := (w \circ u, s \circ v)$.

Easy to prove is then the following

PROPOSITION 1.5.3. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be categories and $f, g \in 2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$. There exists a unique category whose object set is $Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g)$, while $(f, g)\langle \delta, \varepsilon \rangle$ is the class of the morphisms from δ to ε , for all $\delta, \varepsilon \in Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g)$, and the morphism composition is the one defined in Def. 1.5.2.

DEFINITION 1.5.4. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be categories, $f, g, h, k \in 2 - \mathbf{C}(\mathbf{A}, \mathbf{B})$, $\alpha, \alpha' \in Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(f, g)$ and $\beta, \beta' \in Mor_{2-\mathbf{C}(\mathbf{A}, \mathbf{B})}(h, k)$, define

$$D_{h, k; f, g}^{\beta, \beta'; \alpha, \alpha'} := \{((w, z); (u, v)) \in (h, k)\langle \beta, \beta' \rangle \times (f, g)\langle \alpha, \alpha' \rangle \mid h \circ v = w \circ g\}.$$

Moreover define the map

$$\star : D_{h,k,f,g}^{\beta,\beta';\alpha,\alpha'} \ni ((w,z), (u,v)) \mapsto (h \circ u, z \circ g).$$

PROPOSITION 1.5.5. Let A, B, C be categories, $f, g, h, k \in 2 - C(A, B)$, $\alpha, \alpha' \in \text{Mor}_{2-C(A,B)}(f, g)$ and $\beta, \beta' \in \text{Mor}_{2-C(A,B)}(h, k)$, then $\star : D_{h,k,f,g}^{\beta,\beta';\alpha,\alpha'} \rightarrow (h \circ f, k \circ g) \langle \beta * \alpha, \beta' * \alpha' \rangle$.

PROOF. Let $((w,z); (u,v)) \in D_{h,k,f,g}^{\beta,\beta';\alpha,\alpha'}$ and $M \in A$ then

$$\begin{aligned} (\beta' * \alpha')_M \circ h(u_M) &= \beta'_{g(M)} \circ h(\alpha'_M \circ u_M) \\ &= \beta'_{g(M)} \circ h(v_M \circ \alpha_M) \\ &= \beta'_{g(M)} \circ w_{g(M)} \circ h(\alpha_M) \\ &= z_{g(M)} \circ \beta_{g(M)} \circ h(\alpha_M) = z_{g(M)} \circ (\beta * \alpha)_M, \end{aligned}$$

where the second equality comes since $(u,v) \in (f,g) \langle \alpha, \alpha' \rangle$, the third one by $h(v_M) = w_{g(M)}$, the fourth one by $(w,z) \in (h,k) \langle \beta, \beta' \rangle$. \square

DEFINITION 1.5.6. Given any object C of \mathbf{Cat} , let $2 - C$ be the subcategory of \mathbf{Cat} , whose object set is $\text{Obj}(\mathbf{Cat})$, and $\text{Mor}_{2-C}(A, B) := 2 - C(A, B)$, for any $A, B \in \mathbf{Cat}$.

PROPOSITION 1.5.7. $2 - C$ is a 2-category such that $\text{Mor}_{\text{Mor}_{2-C}(A,B)}(f, g)$ is a category for all $f, g \in \text{Mor}_{2-C}(A, B)$ and $A, B \in 2 - C$.

PROOF. The second morphism composition law $*$ in it is an internal law since [3, Prp. 1.3.4], then the first sentence of the statement follows since \mathbf{Cat} is a 2-category, the second one follows since Prp. 1.5.3. \square

1.5.2. Fibered category of connectors. Here we apply the results of section 1.5.1 to the case $C = \mathfrak{dp}$. Since Cor. 1.4.5 \mathfrak{dp} is an object of \mathbf{Cat} . Hence $2 - \mathfrak{dp}$ is well-set according Def. 1.5.6, and it is a 2-category since Prp. 1.5.7, thus we can set the following

DEFINITION 1.5.8 (Fibered category of species). Define \mathfrak{Sp} the fibered category over $2 - \mathfrak{dp}$ such that for all $\mathfrak{D} \in 2 - \mathfrak{dp}$

$$\mathfrak{Sp}(\mathfrak{D}) = 2 - \mathfrak{dp}(\mathfrak{D}, \mathfrak{Ch}\mathfrak{dv}),$$

moreover set

$$\mathfrak{Sp}_* := \{(\mathbf{a}, \mathbf{b}) \in \mathfrak{Sp} \times \mathfrak{Sp} \mid d(\mathbf{a}) = d(\mathbf{b})\}.$$

Notice that since Thm. 1.4.20 we have that $\Psi \circ \text{Fct}(\mathfrak{D}, \mathfrak{dp}) \subset \mathfrak{Sp}(\mathfrak{D})$.

DEFINITION 1.5.9 (Fibered category of connectors). Let \mathfrak{Cnt} be the fibered category over \mathfrak{Sp}_* such that for all $(\mathbf{a}, \mathbf{b}) \in \mathfrak{Sp}_*$

$$\mathfrak{Cnt}(\mathbf{a}, \mathbf{b}) := \text{Mor}_{\mathfrak{Sp}(d(\mathbf{a}))}(\mathbf{a}, \mathbf{b}),$$

with the category structure described in Prp. 1.5.3, called the category of connectors from $\mathfrak{s}(\mathbf{a})$ to $\mathfrak{s}(\mathbf{b})$. Define for all $\mathbf{a} \in \mathfrak{Sp}$ the full subcategory $\mathfrak{Cnt}_{\mathbf{a}}$ of \mathfrak{Cnt} such that $\text{Obj}(\mathfrak{Cnt}_{\mathbf{a}}) = \{\mathbf{T} \in \mathfrak{Cnt} \mid d(\mathbf{T}) = \mathbf{a}\}$, and let \mathfrak{Sct} be the fibered category over \mathfrak{Sp} such that for all $\mathbf{a} \in \mathfrak{Sp}$ we have that $\mathfrak{Sct}(\mathbf{a}) = \mathfrak{Cnt}(\mathbf{a}, \mathbf{a})$, called the category of sectors of $\mathfrak{s}(\mathbf{a})$.

To our point of view some structural data of the category of unitary net representations defined by Brunetti and Ruzzi in [9] seem to present analogies with the general construction in [17, Def. 1.2.5], permitting to consider any $\mathfrak{Sct}(\mathfrak{a})$ the dynamically oriented extension of the construction in [9].

DEFINITION 1.5.10 (Experimental settings generated by \mathfrak{Cnt}). *Let \mathfrak{ex} be the fibered set over \mathfrak{Cnt} such that for all $T \in \mathfrak{Cnt}$*

$$\mathfrak{ex}(T) = \{T[\mathfrak{Q}, t] \mid \mathfrak{Q} \in \text{Exp}(c(T)), t \in \Gamma(\mathfrak{Q}, T)\},$$

called the collection of experimental settings generated by $s(T)$, well-set since $\mathfrak{ex}(T) \subseteq \text{Exp}(d(T))$ by Thm. 1.4.46.

REMARK 1.5.11. Since for any $\mathfrak{a} \in \mathfrak{Sp}$ and any $\mathfrak{Q} = (\mathfrak{S}, R) \in \text{Exp}(\mathfrak{a})$ the map on $\text{Obj}(d(\mathfrak{a}))$, assigning to any object M the identity map on R_M , belongs to $\Gamma(\mathfrak{Q}, 1_{\mathfrak{a}})$ we have that that $\mathfrak{ex}(1_{\mathfrak{a}}) = \text{Exp}(\mathfrak{a})$, i.e. the collection of experimental settings generated by the identity morphism of any species is the maximal one, namely equals the whole set of experimental settings of that species.

Next we use Cor. 1.4.48 in order to relate any morphism between connectors to the experimental settings they generate.

PROPOSITION 1.5.12. *Let $(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{Sp}_*$, $S, T \in \mathfrak{Cnt}(\mathfrak{a}, \mathfrak{b})$, then*

- (1) $\text{Mor}_{\mathfrak{Cnt}(\mathfrak{a}, \mathfrak{b})}(S, T) = \{(u, v) \in \mathfrak{Sct}(\mathfrak{a}) \times \mathfrak{Sct}(\mathfrak{b}) \mid T \circ u = v \circ S\}.$
- (2) *Let $\mathfrak{Q} \in \text{Exp}(\mathfrak{b})$ and $(u, v) \in \text{Mor}_{\mathfrak{Cnt}(\mathfrak{a}, \mathfrak{b})}(S, T)$. If there exist $r \in \Gamma(\mathfrak{Q}, T)$, $q \in \Gamma(T[\mathfrak{Q}, r], u)$ and $l \in \Gamma(\mathfrak{Q}, v)$, $n \in \Gamma(v[\mathfrak{Q}, l], S)$ such that $r \circ q = l \circ n$, then*

$$u[T[\mathfrak{Q}, r], q] = S[v[\mathfrak{Q}, l], n].$$

PROOF. St. (1) is trivial, st. (2) follows by st. (1) and Cor. 1.4.48. □

REMARK 1.5.13. Notice that $u[T[\mathfrak{Q}, r], q] \in \mathfrak{ex}(S)$, namely any morphism from the connector S to the connector T , induces a map from a subset of $\mathfrak{ex}(T)$ to $\mathfrak{ex}(S)$, representing the direct empirical meaning of morphisms between connectors. Even the existence of an isomorphism between connectors does not imply the equality of the collections of experimental settings they generate. This consideration makes unfeasible the attempt to generalize in our context the procedure of passing to quotient with respect to unitary sectors performed in the DHR analysis of algebraic covariant sectors. Only by choosing suitable species \mathfrak{a} and restricting to a specific subcategory say $\Delta(\mathfrak{a})$ of $\mathfrak{Sct}(\mathfrak{a})$ we obtain $\mathfrak{ex}(S) = \mathfrak{ex}(T)$ for any $S, T \in \Delta(\mathfrak{a})$ such that $\text{Inv}_{\Delta(\mathfrak{a})}(S, T) \neq \emptyset$. The reason inherits by the fact that in our framework different connectors, or even sectors, in general generate quite different experimental settings. Because of this rather than seeking for an equivalence of the category of some known mathematical structure with a distinct subcategory of equivalence classes of sectors of some suitable species; it is preferable to establish an equivalence between a suitable 2-category and $2 - \mathfrak{dp}$. This in line with our original purpose of showing that there are properties of invariance concerning *diverse* species, an

emblematic case is the equiformity principle between a classical and a quantum species, we shall return to this point.

Next we present some easy to prove algebraic properties of \mathfrak{Cnt} .

PROPOSITION 1.5.14. (1) $(\forall T, S \in \mathfrak{Cnt})((T, S) \in \text{Dom}(\circ) \Rightarrow T \circ S \in \mathfrak{Cnt})$;
 (2) $(\forall T \in \mathfrak{Cnt})(\forall L \in 2 - \text{cell}(2 - \mathfrak{dp}))((T, L) \in \text{Dom}(\ast) \Rightarrow T \ast L \in \mathfrak{Cnt})$;
 (3) $\text{Mor}_{\mathfrak{Sp}(\mathfrak{Chdd})} \ast \mathfrak{Cnt} \subset \mathfrak{Cnt}$.

1.5.3. Physical posit. Here we describe the physical interpretation of $2 - \mathfrak{dp}$, and in particular that of \mathfrak{Cnt} . Thm. 1.4.46(5) permits the following

DEFINITION 1.5.15 (Equiformity principle of $\mathfrak{s}(T)$ and interpretations). *Let*

$$\mathcal{E}, \mathcal{E}_s \in \prod_{T \in \mathfrak{Cnt}} \prod_{Q \in \text{Exp}(c(T))} \prod_{t \in \Gamma(Q, T)} \text{set},$$

such that

$$\begin{aligned} \mathcal{E}(T)(Q)(t) &:= \{\text{equalities in Prp. 1.4.34 for } \mathfrak{C}^a = T[Q, t], \mathfrak{C}^b = Q\}, \\ \mathcal{E}_s(T)(Q)(t) &:= \{\mathfrak{s}(x) \mid x \in \mathcal{E}(T)(Q)(t)\}. \end{aligned}$$

Let

$$\mathcal{X}_s \in \prod_{T \in \mathfrak{Cnt}} \prod_{Q \in \text{ex}(T)} \text{set},$$

$$\mathcal{X}_s(T)(Q) := \mathfrak{s}(Q).$$

Finally for any $\mathbf{a} \in \mathfrak{Sp}$ set $\mathcal{Z}(\mathbf{a}) := \mathcal{E} \upharpoonright \mathfrak{Ct}(\mathbf{a})$, $\mathcal{Z}_s(\mathbf{a}) := \mathcal{E}_s \upharpoonright \mathfrak{Ct}(\mathbf{a})$.

Essentially $\mathcal{E}_s(T)(Q)(t)$ is a collection of sentences of the sort stated in Prp. 1.4.35. The next reflects our physical interpretation of $2 - \mathfrak{dp}$ which by matter-of-fact represents a paradigm.

POSIT 1.5.16 (Physical reality is $2 - \mathfrak{dp}$). *Our posit consists in what follows*

- (1) $2 - \mathfrak{dp}$ is the structure of the physical reality;
- (2) \mathfrak{Cnt} as a structure and as a collection is the only empirically testable part of $2 - \mathfrak{dp}$;
- (3) each connector of \mathfrak{Cnt} admits an empirical representation and a physical interpretation;
- (4) for any $T \in \mathfrak{Cnt}$
 - (a) $(\mathcal{E}(T), \text{ex}(T))$ is the empirical representation of $\mathfrak{s}(T)$,
 - (b) $(\mathcal{E}_s(T), \mathcal{X}_s(T))$ is the physical interpretation of $\mathfrak{s}(T)$;
- (5) The couple of results: charge composition of connectors Cor. 1.4.48, applied to elements of \mathfrak{Cnt} , and charge transfer Cor. 1.4.53, applied to elements of \mathfrak{Cnt} and to 2 -cells of $2 - \mathfrak{dp}$, is the empirical representation of the algebraic structure of \mathfrak{Cnt} described in Prp. 1.5.14, whose physical interpretation is in terms of the semantics $(\mathfrak{s}, \mathfrak{u})$.

REMARK 1.5.17 (Empirical representation and physical interpretation of species). Since Posit 1.5.16 it follows that for each $\mathbf{a} \in \mathfrak{Sp}$

- (1) $(\mathcal{Z}(\mathbf{a}), \text{ex}(1_{\mathbf{a}}))$ is the empirical representation of $\mathfrak{s}(\mathbf{a})$;
- (2) $(\mathcal{Z}_s(\mathbf{a}), \mathcal{X}_s(1_{\mathbf{a}}))$ is the physical interpretation of $\mathfrak{s}(\mathbf{a})$.

We recall that $\mathfrak{ex}(1_a) = \text{Exp}(\mathfrak{a})$.

CONVENTION 1.5.18. Let $T \in \mathfrak{Cnt}$ and $\mathfrak{a} \in \mathfrak{Sp}$, we convey to call

- (1) $\mathcal{E}(T)$ the equiformity principle of $\mathfrak{s}(T)$;
- (2) $\mathcal{E}_s(T)$ the interpretation of $\mathcal{E}(T)$;
- (3) $\mathcal{E}(T)(\mathfrak{Q})(s)$ the equiformity principle of $\mathfrak{s}(T)$ evaluated on $\mathfrak{s}(T[\mathfrak{Q}, s])$ and $\mathfrak{s}(\mathfrak{Q})$;
- (4) $\mathcal{E}_s(T)(\mathfrak{Q})(s)$ the interpretation of $\mathcal{E}(T)(\mathfrak{Q})(s)$;
- (5) $\mathcal{X}_s(T)$ the interpretation of $\mathfrak{ex}(T)$;
- (6) $\mathcal{Z}(\mathfrak{a})$ the equiformity principle of $\mathfrak{s}(\mathfrak{a})$;
- (7) $\mathcal{Z}_s(\mathfrak{a})$ the interpretation of $\mathcal{Z}(\mathfrak{a})$.

REMARK 1.5.19. We have that

- (1) the empirical representation of $\mathfrak{s}(T)$ consists in the couple of the equiformity principle of $\mathfrak{s}(T)$ and the collection of the experimental settings of $\mathfrak{s}(T)$;
- (2) the equiformity principle of $\mathfrak{s}(\mathfrak{a})$ is the map mapping any sector T of $\mathfrak{s}(\mathfrak{a})$ to the equiformity principle of $\mathfrak{s}(T)$;
- (3) the interpretation of the equiformity principle of $\mathfrak{s}(\mathfrak{a})$ is the map mapping any sector T of $\mathfrak{s}(\mathfrak{a})$ to the interpretation of the equiformity principle of $\mathfrak{s}(T)$;
- (4) $\mathcal{X}_s(1_a)$ is the interpretation of $\text{Exp}(\mathfrak{a})$.

Cor. 1.4.48 permits the following

DEFINITION 1.5.20 (Experimental settings generated by connectors and a vacuum connector). Let $\pi \in \mathfrak{Cnt}$, $\mathfrak{a} \in \mathfrak{Sp}$, $\mathfrak{Q} \in \text{Exp}(c(\pi))$ and $r \in \Gamma(\mathfrak{Q}, \pi)$ set

$$\mathfrak{Ex}(\mathfrak{a}, \pi, \mathfrak{Q}, r) := \{(\pi \circ \mathbf{S})[\mathfrak{Q}, r \circ q] \mid \mathbf{S} \in \mathfrak{Cnt}_a, c(\mathbf{S}) = d(\pi), q \in \Gamma(\pi[\mathfrak{Q}, r], \mathbf{S})\}.$$

Notice that $\mathfrak{Ex}(\mathfrak{a}, \pi, \mathfrak{Q}, r) \subseteq \text{Exp}(\mathfrak{a})$.

COROLLARY 1.5.21. Let $\mathfrak{a} \in \mathfrak{Sp}$, $\pi \in \mathfrak{Cnt}$, $\mathfrak{Q} \in \text{Exp}(c(\pi))$ and $r \in \Gamma(\mathfrak{Q}, \pi)$. Thus

$$\mathfrak{Ex}(\mathfrak{a}, \pi, \mathfrak{Q}, r) = \{\mathbf{S}[\pi[\mathfrak{Q}, r], q] \mid \mathbf{S} \in \mathfrak{Cnt}_a, c(\mathbf{S}) = d(\pi), q \in \Gamma(\pi[\mathfrak{Q}, r], \mathbf{S})\},$$

in particular if $\pi \in \mathfrak{Cnt}_a$, then

$$\mathfrak{Ex}(\mathfrak{a}, \pi, \mathfrak{Q}, r) = \{\mathbf{S}[\pi[\mathfrak{Q}, r], q] \mid \mathbf{S} \in \mathfrak{Sct}(\mathfrak{a}), q \in \Gamma(\pi[\mathfrak{Q}, r], \mathbf{S})\}.$$

PROOF. Since Cor. 1.4.48. □

1.6. Gravity species

In section 1.6.1 we provide, in the fourth main result of the paper, the construction of the n -dimensional classical gravity species \mathfrak{a}^n and then we establish that the equivalence principle of general relativity emerges as equiformity principle of the connector canonically associated with \mathfrak{a}^n . By using \mathfrak{a}^n as a model in section 1.6.2 we define the collection of n -dimensional gravity species in particular diverse subcollections of quantum gravity species. The equiformity principle of a connector from \mathfrak{a}^n to a strict quantum gravity species will provide a quantum realization of the velocity of maximal integral curves of complete vector fields on spacetimes. Applied to Robertson-Walker

spacetimes we establish that the Hubble parameter, the acceleration of the scale function and new constraints for its positivity over a subset of the range of the galactic time of a geodesic α , are expressed in terms of a quantum realization of the velocity of α . As a remarkable result we obtain that the existence of a connector from \mathfrak{a}^4 to a 4-dimensional strict quantum gravity species satisfying these constraints implies the positivity of the acceleration and renders the dark energy hypothesis inessential.

NOTATION 1.6.1. Here manifold means second countable finite dimensional smooth manifold. For any manifold M let $\mathcal{A}(M) := \mathcal{C}^\infty(M, \mathbb{C})$ the complex $*$ -algebra of smooth maps on M at values in \mathbb{C} with \mathbb{C} provided by the pullback of the standard smooth structure on \mathbb{R}^2 . Any open set X of M is tacitly considered as a submanifold of M , so $\mathcal{A}(X)$ makes sense. We identify $\mathcal{A}(M)_{ob}$ with the real linear space underlying $\mathfrak{F}(M) := \mathcal{C}^\infty(M)$, the space of real valued smooth maps on M . For any $p \in M$ let δ_p^M also denoted by ω_p be the Dirac distribution on p , namely the map on $\mathcal{A}(M)$ such that $\delta_p^M(f) = f(p)$. Any open set of M is tacitly considered as an open submanifold of M . Let $\mathfrak{T}_s^r(M)$ denote the (real) linear space of (r, s) tensor fields on M with $(r, s) \in \{0, \dots, n\}^2$, $n = \dim(M)$. Let ξ be any coordinate system on $U \subset M$, $A \in \mathfrak{T}_s^r(M)$, $i \in \{0, \dots, n-1\}^r$ and $j \in \{0, \dots, n-1\}^s$. Then let x_j^ξ and ∂_ξ^j denote the j th coordinate function and vector field of ξ respectively, and $A_{j_1, \dots, j_s}^{i_1, \dots, i_r, \xi} \in \mathcal{C}^\infty(U)$ denote the $(i_1, \dots, i_r; j_1, \dots, j_s)$ component of A relative to ξ , where the case $(r, s) = (0, 0)$ has to be understood as $A \upharpoonright U$; only when the coordinate system ξ involved is known, we often remove the symbol ξ . Let $\mathfrak{X}(M)$ denote the linear space of vector fields on M , often we use the standard identification $\mathfrak{X}(M)$ with the derivations on M , i.e. $W'(h) \in \mathfrak{F}(M)$ such that $W'(h)(q) := W(q)(h)$, for all $W \in \mathfrak{X}(M)$, $h \in \mathfrak{F}(M)$ and $q \in M$, [25, p. 12]. Often we use also the identification $\mathfrak{X}(M) = \mathfrak{T}_0^1(M)$. Let M, N be manifolds, $\phi : M \rightarrow N$ smooth, then $d\phi : TM \rightarrow TN$ such that $d\phi(v)(h) := v(h \circ \phi)$, for all $v \in TM$ and $h \in \mathfrak{F}(N)$, [25, p. 9]. If $A \in \mathfrak{T}_s^0(N)$ with $s \in \mathbb{N}$, then $\phi^*A \in \mathfrak{T}_s^0(M)$ is the pullback of A by ϕ defined in [25, p. 42, Def. 8]. For any manifold M and $V, W \in \mathfrak{X}(M)$ we have that $[V, W] \in \mathfrak{X}(M)$ such that $[V, W](p)(f) = V(p)(W(f)) - W(p)(V(f))$ for all $f \in \mathfrak{F}(M)$ and $p \in M$, [25, p. 13]. If $\phi : M \rightarrow N$ is a smooth map, $A \in \mathfrak{X}(M)$ and $B \in \mathfrak{X}(N)$, then by following the definition given in [18, p.182] or [25, Def.1.20] we say that A and B are ϕ -related if $d(\phi) \circ A = B \circ \phi$, if ϕ is a diffeomorphism, then for any vector field A on M we let $d\phi A$ denote the unique vector field on N ϕ -related to A , namely $d\phi A := (d\phi) \circ A \circ \phi^{-1}$, [25, pg. 14]. For any complete smooth vector field V on M let $\mathfrak{V}^V : \mathbb{R} \rightarrow \text{Diff}(M)$ the flow of V , hence $\mathfrak{V}^V(\tau)(p) = \alpha_p^V(\tau)$, with $p \in M$ and $\tau \in \mathbb{R}$ and α_p^V the unique inextendible integral curve of V such that $\alpha(0) = p$. For any smooth map $\phi : M \rightarrow N$ where N is a manifold, we let ϕ^* denote the usual pullback, so in particular $\phi^* : \mathcal{C}^\infty(N) \rightarrow \mathcal{C}^\infty(M)$ such that $f \mapsto f \circ \phi$. In order to avoid conflict with the notation used in section 1.2 for general invertible morphisms in a category, we convey that ϕ^* always refers to the pullback of ϕ whenever ϕ is a smooth map between manifolds. If the contrary is not asserted, let π denote the projection map from TM to M . We adopt for semi-Riemannian geometry the conventions in [25], in particular let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold then we let $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ or simply $\langle \cdot, \cdot \rangle$ denote g and whenever it does not cause

confusion let $D^{\mathcal{M}}$ be the Levi-Civita connection of (M, g) . We let $\mathfrak{X}(\mathcal{M})$ denote $\mathfrak{X}(M)$, let $g_{\mathcal{M}}$ denote g and let $\phi : \mathcal{M} \rightarrow \mathcal{N}$ denote $\phi : M \rightarrow N$ for any semi-Riemannian manifold $\mathcal{N} = (N, h)$. We say $U \in \mathfrak{X}(\mathcal{M})$ to be geodesic if $D_U^{\mathcal{M}}U = 0$. Notice that the integral curves of a geodesic vector field are geodesic curves. $v \in T_p M$ is timelike if $g_p(v, v) < 0$, while U is timelike if $\langle U, U \rangle_{\mathcal{M}} < 0$. If $X, Y \in \mathfrak{X}(\mathcal{M})$, then $X \perp Y$ means $\langle X, Y \rangle_{\mathcal{M}} = 0$, for any $U \in \mathfrak{X}(\mathcal{M})$ let U^\perp the set of all $Z \in \mathfrak{X}(\mathcal{M})$ such that $X \perp U$. Let $R^{\mathcal{M}}$ and $Ric^{\mathcal{M}}$ denote the Riemannian curvature and Ricci tensor of \mathcal{M} respectively [25, Def. 3.35, Def. 3.53], Assume \mathcal{M} and \mathcal{N} be spacetimes see [25, p. 163] except the restriction of dimension equal to 4. If $\mathfrak{T}^{\mathcal{M}}$ and $\mathfrak{T}^{\mathcal{N}}$ are the timelike smooth vector fields of \mathcal{M} and \mathcal{N} respectively determining the orientations of \mathcal{M} and \mathcal{N} according to [25, Lemma 5.32], then we say that ϕ preserves the orientation, iff $\mathfrak{T}^{\mathcal{M}}$ and $\mathfrak{T}^{\mathcal{N}}$ are ϕ -related. $S^{\mathcal{M}}$ is the scalar curvature of \mathcal{M} , [25, Def. 3.53], and $G^{\mathcal{M}}$ be the Einstein gravitational tensor [25, Def. 12.1]. Let $T^{\mathcal{M}}$ be the stress-energy tensor associated with \mathcal{M} namely such that the Einstein equation $G^{\mathcal{M}} = 8\pi T^{\mathcal{M}}$ holds. According [25, Def. 12.4] we have that (U, ρ, p) is a perfect fluid on \mathcal{N} if $U \in \mathfrak{X}(\mathcal{N})$ timelike future-pointing unit, $\rho, p \in \mathfrak{F}(\mathcal{N})$, and for all $X, Y \in U^\perp$ we have $T^{\mathcal{N}}(U, U) = \rho$, $T^{\mathcal{N}}(X, U) = T^{\mathcal{N}}(U, X) = 0$, and $T^{\mathcal{N}}(X, Y) = p\langle X, Y \rangle_{\mathcal{N}}$. Let $n \in \mathbb{N}$ and $\{\varepsilon_\mu\}_{\mu=0}^n$ be the standard basis in \mathbb{R}^{n+1} , we sometime let $x^\mu = x(\mu)$ for $x \in \mathbb{R}^{n+1}$, let $\text{Pr}_\mu : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be $x \mapsto x(\mu)$ and $\iota_\mu : \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ be such that $\text{Pr}_\nu \circ \iota_\mu = 1_{\mathbb{R}} \delta_{\mu, \nu}$, for all $\mu, \nu \in \{0, \dots, n\}$. For any linear operator L on \mathbb{R}^{n+1} let $\mathbf{m}(L)$ be the matrix of L w.r.t. the canonical basis of \mathbb{R}^{n+1} , i.e. $\sum_{\nu=0}^n \mathbf{m}(L)_{\mu\nu} \varepsilon_\nu = L(\varepsilon_\mu)$, for all $\mu, \nu \in \{0, \dots, n\}$. Let \mathbb{R}_1^4 be the Minkowski 4-space, while for any Minkowski spacetime M let $\text{Lor}(M)$ be the set of the Lorentz coordinate systems in M , see [25, p.163-164, Def. 8 and 11]. Let $\mathcal{P}^+ = \mathbb{R}^4 \rtimes_\eta SL(2, \mathbb{C})$, where η is the standard action of $SL(2, \mathbb{C})$ on \mathbb{R}^4 , and let j_1 and j_2 be the canonical injection of \mathbb{R}^4 and $SL(2, \mathbb{C})$ into \mathcal{P}^+ respectively, see [30] and reference therein. Let ρ be the action of \mathcal{P}^+ on \mathbb{R}^4 such that $\rho(x, \Lambda)y := x + \eta(\Lambda)y$, for any $\Lambda \in SL(2, \mathbb{C})$ and $x, y \in \mathbb{R}^4$. Let LS denote the category of real linear spaces.

1.6.1. n -dimensional classical gravity species. After preparatory lemmata, the first result of this section is **Thm. 1.6.12** where we prove that suitable ϕ preserve some properties of ϕ -related vector fields of semi-Riemannian manifolds, such as the property of being geodesic and, in case of spacetimes, of being the support field of a perfect fluid. In Def. 1.6.15 we introduce the category \mathbf{vf}_0 whose object set is the collection of couples of manifolds and complete smooth vector fields on them, with smooth open maps relating the vector fields as morphisms. In Def. 1.6.16 we provide the definition of the category \mathbf{St}_n of n -dimensional spacetimes and observer fields, naturally embedded into \mathbf{vf}_0 . In Def. 1.6.18 we define the collection of the relevant topologies on the algebras $\mathcal{A}(M)$ for all manifolds M , employed in Prp. 1.6.22 to state that with any object of \mathbf{vf}_0 remains associated a dynamical pattern, namely an object of the category \mathbf{dp} , whose dynamical category we construct in Prp. 1.6.17. Prp. 1.6.22 is the essential step toward **Thm. 1.6.24**, the main result of this section, where we construct a species on \mathbf{vf}_0 , namely a functor from \mathbf{vf}_0 to the category $\mathbf{Ch\ddot{v}}$. As a consequence we obtain in Cor. 1.6.26 the n -dimensional classical gravity species $\mathbf{a}^n \in \mathfrak{Sp}(\mathbf{St}_n)$. Then we state in **Cor. 1.6.28** that the equivalence

principle of general relativity is the equiformity principle of the connector canonically associated with \mathfrak{a}^n .

LEMMA 1.6.2. *Let M_1, M_2 be manifolds, $\phi : M_1 \rightarrow M_2$ smooth, and $V_i, W_i \in \mathfrak{X}(M_i)$ for any $i \in \{1, 2\}$. If V_1 and V_2 as well W_1 and W_2 are ϕ -related then $[V_1, W_1]$ and $[V_2, W_2]$ are ϕ -related.*

PROOF. Let $p \in M_1$ and $f \in \mathfrak{F}(M_2)$, then

$$\begin{aligned} ([V_2, W_2] \circ \phi)(p)(f) &= (V_2 \circ \phi)(p)(W_2(f)) - (W_2 \circ \phi)(p)(V_2(f)) \\ &= (d\phi \circ V_1)(p)(W_2(f)) - (d\phi \circ W_1)(p)(V_2(f)) \\ &= V_1(p)(W_2(f) \circ \phi) - W_1(p)(V_2(f) \circ \phi) \\ &= V_1(p)((W_2 \circ \phi)(\cdot)(f)) - W_1(p)((V_2 \circ \phi)(\cdot)(f)) \\ &= V_1(p)((d\phi \circ W_1)(\cdot)(f)) - W_1(p)((d\phi \circ V_1)(\cdot)(f)) \\ &= V_1(p)(W_1(f \circ \phi)) - W_1(p)(V_1(f \circ \phi)) \\ &= [V_1, W_1](p)(f \circ \phi) = (d\phi \circ [V_1, W_1])(p)(f). \end{aligned}$$

□

PROPOSITION 1.6.3. *Let M, N be manifolds, $\phi : M \rightarrow N$ smooth, $A \in \mathfrak{T}_s^0(N)$ with $s \in \mathbb{N}$, $X_r \in \mathfrak{X}(M)$, $Y_r \in \mathfrak{X}(N)$ such that X_r and Y_r are ϕ -related for any $r \in \{1, \dots, s\}$. Then $(\phi^*A)(X_1, \dots, X_s) = A(Y_1, \dots, Y_s) \circ \phi$. Let $B \in \mathfrak{T}_s^0(M)$ and assume that ϕ has dense range. If for any $(X_1, \dots, X_s) \in \mathfrak{X}(M)^s$ there exists $(Y_1, \dots, Y_s) \in \mathfrak{X}(N)^s$, such that $B(X_1, \dots, X_s) = A(Y_1, \dots, Y_s) \circ \phi$ and X_r and Y_r are ϕ -related for any $r \in \{1, \dots, s\}$, then $B = \phi^*A$.*

PROOF. Let $p \in M$ then

$$\begin{aligned} (\phi^*A)(X_1, \dots, X_s)(p) &= (\phi^*A)(p)(X_1(p), \dots, X_s(p)) \\ &= A(\phi(p))((d\phi \circ X_1)(p), \dots, (d\phi \circ X_s)(p)) \\ (25) \quad &= A(\phi(p))((Y_1 \circ \phi)(p), \dots, (Y_s \circ \phi)(p)) \\ &= (A(Y_1, \dots, Y_s) \circ \phi)(p). \end{aligned}$$

The second sentence of the statement follows by the first one and since $\mathfrak{F}(M) \subset \mathcal{C}(M)$. □

Till Thm. 1.6.12 let \mathcal{M}, \mathcal{N} be semi-Riemannian manifolds, $m = \dim(\mathcal{M})$, $n = \dim(\mathcal{N})$ and $\phi : \mathcal{M} \rightarrow \mathcal{N}$ smooth such that $g_{\mathcal{M}} = \phi^*g_{\mathcal{N}}$.

LEMMA 1.6.4. *Let $U, V, Y \in \mathfrak{X}(\mathcal{M})$ and $\mathcal{U}, \mathcal{V}, \mathcal{Y} \in \mathfrak{X}(\mathcal{N})$, such that U and \mathcal{U} , V and \mathcal{V} , and Y and \mathcal{Y} are ϕ -related respectively. Thus $\langle D_{\mathcal{U}}^{\mathcal{M}}V, Y \rangle_{\mathcal{M}} = \langle D_{\mathcal{U}}^{\mathcal{N}}\mathcal{V}, \mathcal{Y} \rangle_{\mathcal{N}} \circ \phi$.*

PROOF. Since Prp. 1.6.3 applied to $A = g_{\mathcal{N}}$ we have $\langle V, Y \rangle_{\mathcal{M}} = \langle \mathcal{V}, \mathcal{Y} \rangle_{\mathcal{N}} \circ \phi$, and $\langle U, V \rangle_{\mathcal{M}} = \langle \mathcal{U}, \mathcal{V} \rangle_{\mathcal{N}} \circ \phi$, while since Prp. 1.6.3 and Lemma 1.6.2 we obtain $\langle U, [V, Y] \rangle_{\mathcal{M}} = \langle \mathcal{U}, [\mathcal{V}, \mathcal{Y}] \rangle_{\mathcal{N}} \circ \phi$, and the statement follows since the Koszul formula [25, p. 61 Thm. 11] applied to $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{N}}$. □

LEMMA 1.6.5. *Let $X, Y, Z, K \in \mathfrak{X}(\mathcal{M})$ and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{K} \in \mathfrak{X}(\mathcal{N})$, such that X and \mathcal{X} , Y and \mathcal{Y} , Z and \mathcal{Z} and K and \mathcal{K} are ϕ -related respectively. Then $\langle R_{XY}^{\mathcal{M}}Z, K \rangle_{\mathcal{M}} = \langle R_{\mathcal{X}\mathcal{Y}}^{\mathcal{N}}\mathcal{Z}, \mathcal{K} \rangle_{\mathcal{N}} \circ \phi$.*

PROOF. Since Lemma 1.6.4 and [25, Lemma 3.35]. \square

LEMMA 1.6.6. Let $\{E_k\}_{k=1}^m$ be a frame field on \mathcal{M} . If the range of ϕ is dense and $\{\mathcal{E}_k\}_{k=1}^m$ is a m -tuple of smooth vector fields on \mathcal{N} such that E_k and \mathcal{E}_k are ϕ -related for all $k \in \{1, \dots, m\}$, then $\{\mathcal{E}_k\}_{k=1}^m$ is an orthogonal system of unit vector fields on \mathcal{N} such that $\langle E_i, E_k \rangle_{\mathcal{M}} = \langle \mathcal{E}_i, \mathcal{E}_k \rangle_{\mathcal{N}} \circ \phi$.

PROOF. Since Prp. 1.6.3 we obtain the equality, the first sentence of the statement then follows by this equality and by the fact that the range of ϕ is dense. \square

DEFINITION 1.6.7. Let the property $\star(\phi)$ be the following proposition $(\forall X \in \mathfrak{X}(\mathcal{M}))(\exists \mathcal{X} \in \mathfrak{X}(\mathcal{N}))(X, \mathcal{X} \text{ are } \phi\text{-related})$. While let the property $\ast(\phi)$ be the following proposition $(\forall \mathcal{X} \in \mathfrak{X}(\mathcal{N}))(\exists X \in \mathfrak{X}(\mathcal{M}))(X, \mathcal{X} \text{ are } \phi\text{-related})$.

Clearly the properties $\star(\phi)$ and $\ast(\phi)$ are satisfied in case ϕ is a diffeomorphism.

LEMMA 1.6.8. Let the property $\star(\phi)$ be satisfied, let the range of ϕ be dense, and let $m = n$. Thus for all $X, Y \in \mathfrak{X}(\mathcal{M})$ and $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(\mathcal{N})$ such that X, \mathcal{X} and Y, \mathcal{Y} are ϕ -related respectively, we have that $\text{Ric}^{\mathcal{M}}(X, Y) = \text{Ric}^{\mathcal{N}}(\mathcal{X}, \mathcal{Y}) \circ \phi$.

PROOF. Let $\{E_k\}_{k=1}^m$ be a frame field on \mathcal{M} and let $\{\mathcal{E}_k\}_{k=1}^m$ be any m -tuple of smooth vector fields on \mathcal{N} such that E_k, \mathcal{E}_k are ϕ -related, existing by the property $\star(\phi)$, thus since Lemma 1.6.6 and the hypothesis $m = n$ we obtain that

$$(26) \quad \{\mathcal{E}_k\}_{k=1}^m \text{ is a frame field on } \mathcal{N}.$$

Next

$$\begin{aligned} \text{Ric}^{\mathcal{M}}(X, Y) &= \sum_{k=1}^m \langle E_k, E_k \rangle_{\mathcal{M}} \langle R_{XE_i}^{\mathcal{M}} Y, E_i \rangle_{\mathcal{M}} \\ &= \sum_{k=1}^m (\langle \mathcal{E}_k, \mathcal{E}_k \rangle_{\mathcal{N}} \circ \phi) \langle R_{\mathcal{X}\mathcal{E}_i}^{\mathcal{N}} \mathcal{Y}, \mathcal{E}_i \rangle_{\mathcal{N}} \circ \phi \\ &= \text{Ric}^{\mathcal{N}}(\mathcal{X}, \mathcal{Y}) \circ \phi, \end{aligned}$$

where the first equality follows since [25, Lemma 3.52], the second one by Lemma 1.6.5 and the equality in Lemma 1.6.6, the third equality follows by [25, Lemma 3.52] and (26). \square

LEMMA 1.6.9. Let ξ be a coordinate system on $U \subseteq N$ and assume that ϕ is a diffeomorphism, then $\partial_j^\xi, \partial_j^{\xi \circ \phi}$ and $(dx_\xi^j)_*, (dx_{\xi \circ \phi}^j)_*$ are $\phi \upharpoonright \phi^{-1}(U)$ -related for any $j \in \{1, \dots, n\}$ respectively, where $(\cdot)_*$ means metrically equivalent vector field.

PROOF. For the coordinate vector fields follows by the definition and [25, Lemma 1.21]. $(dx_\xi^j)_*, (dx_{\xi \circ \phi}^j)_*$ are $\phi \upharpoonright \phi^{-1}(U)$ -related since the formula in the proof of [25, Prp. 3.10] applied to $\theta = dx_\xi^j$, Lemma 1.6.3 applied for $A = g_N$ and to the coordinate vector fields just now proved to be ϕ -related. \square

LEMMA 1.6.10. Let ξ be a coordinate system on $U \subseteq N$ and assume that ϕ is a diffeomorphism, then $S^{\mathcal{M}} = S^{\mathcal{N}} \circ \phi$.

PROOF. We have

$$\begin{aligned}
 C(\uparrow_1^1 Ric^{\mathcal{M}}) &= \sum_{i=1}^n \uparrow_1^1 Ric^{\mathcal{M}}(dx_{\xi \circ \phi}^i, \partial_i^{\xi \circ \phi}) \\
 &= \sum_{i=1}^n Ric^{\mathcal{M}}((dx_{\xi \circ \phi}^i)_*, \partial_i^{\xi \circ \phi}) \\
 &= \sum_{i=1}^n Ric^{\mathcal{N}}((dx_{\xi}^i)_*, \partial_i^{\xi}) \circ \phi \\
 &= C(\uparrow_1^1 Ric^{\mathcal{N}}) \circ \phi.
 \end{aligned}$$

where the first and the fourth equalities follow since the formula in the proof of [25, Lemma 2.6] applied to $\xi \circ \phi, S^{\mathcal{M}}$ and $\xi, S^{\mathcal{N}}$ respectively, the second equality follows since the definition of the metric contraction, while the third one follows by Lemma 1.6.9 and Lemma 1.6.8. \square

LEMMA 1.6.11. *If ϕ is a diffeomorphism, then for any $X, Y \in \mathfrak{X}(M)$ we have $G^{\mathcal{M}}(X, Y) = G^{\mathcal{N}}(d\phi X, d\phi Y) \circ \phi$.*

PROOF. Since Lemma 1.6.8, Lemma 1.6.10 and Prp. 1.6.3 applied to $A = g_{\mathcal{N}}$. \square

Now we can state the following

THEOREM 1.6.12. *Let \mathcal{M}, \mathcal{N} be semi-Riemannian manifolds, $\phi : \mathcal{M} \rightarrow \mathcal{N}$ smooth, $U \in \mathfrak{X}(\mathcal{M})$ and $\mathcal{U} \in \mathfrak{X}(\mathcal{N})$ such that $g_{\mathcal{M}} = \phi^* g_{\mathcal{N}}$ and U and \mathcal{U} are ϕ -related. Then*

- (1) *If the property $\star(\phi)$ is satisfied, then (1a) \Rightarrow (1b) where*
 - (a) \mathcal{U} geodesic
 - (b) U geodesic.
- (2) *If the property $*$ (ϕ) is satisfied and ϕ has a dense range then (1b) \Rightarrow (1a), in particular if ϕ is a diffeomorphism, then (1a) \Leftrightarrow (1b).*
- (3) *If \mathcal{M} and \mathcal{N} are spacetimes and ϕ is a diffeomorphism preserving the orientation, then (3a) \Leftrightarrow (3b) where*
 - (a) (\mathcal{U}, ρ, p) perfect fluid on \mathcal{N} ,
 - (b) $(U, \phi_* \rho, \phi_* p)$ perfect fluid on \mathcal{M} .

PROOF. St. (1) follows since Lemma 1.6.4 and [25, p. 60, Prp. 10], st. (2) follows since Lemma 1.6.4, [25, p. 60, Prp. 10], and since $\langle \cdot, \cdot \rangle_{\mathcal{N}}$ takes values in $\mathfrak{F}(\mathcal{N}) \subset \mathcal{C}(\mathcal{N})$. Assume that (\mathcal{U}, ρ, p) perfect fluid on \mathcal{N} , then since Lemma 1.6.11 we obtain

$$\begin{aligned}
 T^{\mathcal{M}}(U, U) &= T^{\mathcal{N}}(\mathcal{U}, \mathcal{U}) \circ \phi \\
 &= \rho \circ \phi = \phi_* \rho.
 \end{aligned}
 \tag{27}$$

Next let $\mathcal{X} \in \mathcal{U}^{\perp}$, thus $d\phi^{-1}\mathcal{X} \in U^{\perp}$ since Prp. 1.6.3 applied to $A = g_{\mathcal{N}}$, hence $d\phi^{-1}\mathcal{U}^{\perp} \subseteq U^{\perp}$, similarly $d\phi U^{\perp} \subseteq \mathcal{U}^{\perp}$, thus

$$d\phi^{-1}\mathcal{U}^{\perp} = U^{\perp}.
 \tag{28}$$

Next let $\mathcal{X}, \mathcal{Y} \in \mathcal{U}^\perp$, then since Lemma 1.6.11 and Prp. 1.6.3 applied to g_N . we obtain

$$\begin{aligned}
 T^{\mathcal{M}}(d\phi^{-1}\mathcal{X}, U) &= T^{\mathcal{N}}(\mathcal{X}, U) \circ \phi = 0, \\
 T^{\mathcal{M}}(U, d\phi^{-1}\mathcal{X}) &= T^{\mathcal{N}}(U, \mathcal{X}) \circ \phi = 0, \\
 (29) \quad T^{\mathcal{M}}(d\phi^{-1}\mathcal{X}, d\phi^{-1}\mathcal{Y}) &= T^{\mathcal{N}}(\mathcal{X}, \mathcal{Y}) \circ \phi \\
 &= (p\langle \mathcal{X}, \mathcal{Y} \rangle_N) \circ \phi \\
 &= (p \circ \phi)\langle \mathcal{X}, \mathcal{Y} \rangle_N \circ \phi \\
 &= \phi_* p\langle d\phi^{-1}\mathcal{X}, d\phi^{-1}\mathcal{Y} \rangle_{\mathcal{M}}.
 \end{aligned}$$

Since (27), (28) and (29) to prove st. (3) it remains to prove that U is timelike future-pointing and unit. U is timelike and unit since Prp. 1.6.3 applied to g_N . By hypothesis ϕ preserves the orientation, which implies since ϕ is a diffeomorphism that if $\mathfrak{T}^{\mathcal{M}}$ and $\mathfrak{T}^{\mathcal{N}}$ are the timelike smooth vector fields of \mathcal{M} and \mathcal{N} respectively determining the orientations of \mathcal{M} and \mathcal{N} according [25, Lemma 5.32], then we require that $\mathfrak{T}^{\mathcal{N}} = d\phi\mathfrak{T}^{\mathcal{M}}$. Therefore U is future-pointing since Prp. 1.6.3 applied to g_N . \square

PROPOSITION 1.6.13. *There exists a unique category \mathbf{vf} such that*

- (1) $Obj(\mathbf{vf}) = \{(M, U) \mid M \text{ is a manifold, } U \in \mathfrak{X}(M) \text{ complete}\};$
- (2) *for all $(M, U), (N, V), (L, W) \in Obj(\mathbf{vf})$, we have*
 - (a) $Mor_{\mathbf{vf}}((M, U), (N, V)) = \{\phi : N \rightarrow M \text{ smooth} \mid V \text{ and } U \text{ are } \phi\text{-related}\};$
 - (b) $\psi \circ_{\mathbf{vf}} \phi = \phi \circ \psi$, for any $\phi \in Mor_{\mathbf{vf}}((M, U), (N, V))$ and $\psi \in Mor_{\mathbf{vf}}((N, V), (L, W))$;
 - (c) $1_{(M, U)} = 1_M$.

PROOF. Since the chain rule, see for example [25, p. 10, Lemma 15], the morphism composition is a well-set internal associative partial operation. The remaining of the statement is easy to prove. \square

REMARK 1.6.14. Since Thm. 1.6.12 we obtain that for any $(M, U), (N, V) \in \mathbf{vf}$ and $\phi \in Mor_{\mathbf{vf}}((M, U), (N, V))$ such that M and N are supports of semi-Riemannian manifolds (M, g_M) and (N, g_N) , $g_N = \phi^*g_M$ and ϕ has dense range, then V geodesic implies U geodesic, if in addition ϕ is a diffeomorphism then V geodesic is equivalent to U geodesic.

DEFINITION 1.6.15. *Let \mathbf{vf}_0 be the subcategory of \mathbf{vf} with the same object set and $Mor_{\mathbf{vf}_0}((M, U), (N, V))$ the subset of the $\phi \in Mor_{\mathbf{vf}}((M, U), (N, V))$ such that ϕ is an open map.*

Since [23, Cor. 2.3.] any submersion is an open map, in particular any diffeomorphism is an open map.

DEFINITION 1.6.16. *For any $n \in \mathbb{Z}_0^+$, let \mathbf{St}_n be the category whose object set is the set of the couples (\mathcal{M}, U) where \mathcal{M} is an n -dimensional connected time-oriented Lorentz manifold and $U \in \mathfrak{X}(\mathcal{M})$ complete, timelike and unit future-pointing, while $Mor_{\mathbf{St}_n}((\mathcal{M}, U), (\mathcal{N}, V))$ is the set of the submersions $\phi : \mathcal{N} \rightarrow \mathcal{M}$ preserving the orientation, such that $\phi^*g_{\mathcal{M}} = g_{\mathcal{N}}$ and V and U are ϕ -related; while the morphism composition is the opposite of map composition. Let ι_n be the couple of maps defined the first on $Obj(\mathbf{St}_n)$ and the second on $Mor(\mathbf{St}_n)$ such that the first one maps any $((M, g), U)$ into (M, U) and the second one any morphism ϕ into itself.*

Clearly ι_n is a functor from \mathbf{St}_n to \mathbf{vf}_0 , In Prp. 1.6.22 we will show that with any object of \mathbf{vf}_0 remains associated a dynamical pattern.

PROPOSITION 1.6.17. *Let M be a manifold and $U \in \mathfrak{X}(M)$ be complete. There exists a unique **top**-quasi enriched category $[M, U]$ such that*

- (1) $\text{Obj}([M, U]) = \{W \mid W \in \text{Op}(M)\};$
- (2) for all $X, Y, Z \in [M, U]$
 - (a) $\text{Mor}_{[M, U]}(X, Y) = \{(X, Y)\} \times \text{mor}_{[M, U]}(X, Y)$, where
$$\text{mor}_{[M, U]}(X, Y) = \{t \in \mathbb{R} \mid \vartheta^U(t)Y \subseteq X\}$$
 - (b) for any $t \in \text{mor}_{[M, U]}(X, Y)$ and $s \in \text{mor}_{[M, U]}(Y, Z)$

$$((Y, Z), s) \circ ((X, Y), t) = ((X, Z), s + t),$$
 - and $((X, X), 0)$ is the identity morphism of X ;
 - (c) $\text{Mor}_{[M, U]}(X, Y)$ is provided by the topology inherited by that in \mathbb{R} .

PROOF. The composition is a well-defined associative partial operation since ϑ^U is a group morphism, and the category is **top**-enriched since the sum in \mathbb{R} is continuous. \square

We shall prove in Prp. 1.6.22 that a dynamical pattern remains associated with any object of \mathbf{vf} , then determining the object map of a functor from \mathbf{vf}_0 to \mathbf{dp} as established in Thm. 1.6.24. Both these results reside in the pertinent choice of a class of topologies of the algebra $\mathcal{A}(M)$. This class of topologies results intrinsically related to the category \mathbf{vf} as we can see in the following

DEFINITION 1.6.18. *We call **vf**-topology any map ξ , defined on $\text{Obj}(\mathbf{vf})$ such that for all $[M, U] \in \mathbf{vf}$ we have*

- (1) $\xi_{[M, U]}$ is a Hausdorff topology on $\mathcal{A}(M)$ providing it with the structure of topological $*$ -algebra;
- (2) $t \mapsto f \circ \vartheta^U(t)$ is a continuous map from \mathbb{R} to $\langle \mathcal{A}(M), \xi_{[M, U]} \rangle$, for all $f \in \mathcal{A}(M)$;
- (3) $f \mapsto f \circ \phi$ is a continuous map from $\langle \mathcal{A}(M), \xi_{[M, U]} \rangle$ to $\langle \mathcal{A}(N), \xi_{[N, V]} \rangle$, for all $[N, V] \in \mathbf{vf}$ and all $\phi \in \text{Mor}_{\mathbf{vf}}([M, U], [N, V])$.

We call ξ locally convex if all its values are locally convex topologies. For any $[M, U] \in \mathbf{vf}$ and any submanifold X of M we let $\langle \mathcal{A}(X), \xi_{[M, U]} \rangle$ denote the algebra $\mathcal{A}(X)$ provided with the topology inherited by $\xi_{[M, U]}$.

Note that Def. 1.6.18(2) is well-set since the composition of smooth maps is smooth.

PROPOSITION 1.6.19. *The map assigning to any object $[M, U] \in \mathbf{vf}$ the pointwise topology on $\mathcal{A}(M)$ is a locally convex **vf**-topology, in particular the collection of **vf**-topologies is nonempty.*

PROOF. Let $f \in \mathcal{A}(M)$ and $p \in M$ we have that $(f \circ \vartheta^U(t))(p) = (f \circ \alpha_p^U(t))(p)$, next α_p^U is continuous so $t \mapsto (f \circ \vartheta^U(t))(p)$ is continuous, hence $t \mapsto (f \circ \vartheta^U(t))$ is continuous in the pointwise topology. The remaining of the statement is easy to prove. \square

REMARK 1.6.20. The map assigning to any object $[M, U] \in \mathbf{vf}$ the Schwartz topology [21, Ch. 4, n° 4.2] on $\mathcal{A}(M)$ satisfies Def. 1.6.18(1,3), presently it is not clear if it satisfies also Def. 1.6.18(2).

CONVENTION 1.6.21. For the remaining of the paper we consider fixed a \mathbf{vf} -topology ξ .

PROPOSITION 1.6.22. *Let M be a manifold and $U \in \mathfrak{X}(M)$ be complete. There exists a unique $F_{[M,U]} \in \mathbf{Fct}_{\text{top}}([M, U], \mathbf{tsa})$ such that for all $X, Y \in [M, U]$, $t \in \text{mor}_{[M,U]}(X, Y)$ and $f \in \mathcal{A}(X)$*

- (1) $F_{[M,U]}(X) = \langle \mathcal{A}(X), \xi_{[M,U]} \rangle$;
- (2) $F_{[M,U]}((X, Y), t)f = f \circ \vartheta^U(t) \upharpoonright Y^{12}$.

PROOF. Let us assume the notation of the statements. The second statement is well-set since $\vartheta^U(t)Y \subseteq X$. Clearly $F_{[M,U]}$ is uniquely determined by the properties in the statements, moreover since Def. 1.6.18(2) and by recalling that for all $\mathcal{X}, \mathcal{Y} \in \mathbf{tsa}$ the space $\text{Mor}_{\mathbf{tsa}}(\mathcal{X}, \mathcal{Y})$ is provided with the topology of simple convergence, we obtain that

$$F_{[M,U]} \in \mathcal{C}(\text{Mor}_{[M,U]}(X, Y), \text{Mor}_{\mathbf{tsa}}(F_{[M,U]}(X), F_{[M,U]}(Y))).$$

Next

$$\begin{aligned} F_{[M,U]}(((Y, Z), s) \circ ((X, Y), t))f &= \\ f \circ \vartheta^U(t + s) \upharpoonright Z &= \\ f \circ (\vartheta^U(t) \upharpoonright Y) \circ \vartheta^U(s) \upharpoonright Z &= \\ (F_{[M,U]}((Y, Z), s) \circ F_{[M,U]}((X, Y), t))f, \end{aligned}$$

and $(F_{[M,U]}((X, X), 0)f = f$. □

REMARK 1.6.23. Prp. 1.6.22 states that $\langle [M, U], F_{[M,U]} \rangle \in \mathfrak{dp}$, see Def. 1.4.1.

Now we are in the position of stating the following

THEOREM 1.6.24. *There exists a unique $\alpha \in \mathbf{Fct}(\mathbf{vf}_0, \mathfrak{dp})$ such that for all $(M, U), (N, V) \in \mathbf{vf}_0$ and $\phi \in \text{Mor}_{\mathbf{vf}_0}((M, U), (N, V))$*

- (1) $\alpha((M, U)) = \langle [M, U], F_{[M,U]} \rangle$,
- (2) $\alpha(\phi) = (f_\phi, T_\phi)$;

where $f_\phi \in \mathbf{Fct}_{\text{top}}([N, V], [M, U])$ and $T_\phi \in \text{Mor}_{\mathbf{Fct}([N,V], \mathbf{tsa})}(F_{[M,U]} \circ f_\phi, F_{[N,V]})$ such that for all $Y, Z \in [N, V]$ and $t \in \text{mor}_{[N,V]}((Y, Z))$

- (1) $f_\phi(Y) = \phi(Y)$;
- (2) $f_\phi((Y, Z), t) = ((\phi(Y), \phi(Z)), t)$;
- (3) $T_\phi(Y) : \mathcal{A}(\phi(Y)) \ni h \mapsto h \circ \phi \upharpoonright Y \in \mathcal{A}(Y)$.

In particular $\Psi \circ \alpha \in \text{Mor}_{2-\mathfrak{dp}}(\mathbf{vf}_0, \mathfrak{Ch}\mathfrak{dv})$, namely $\Psi \circ \alpha \in \mathfrak{Sp}(\mathbf{vf}_0)$ where Ψ is the functor in Thm. 1.4.20.

¹²we convey to simplify the notation letting $F_{[M,U]}((X, Y), t)$ denote $F_{[M,U]}(((X, Y), t))$.

PROOF. Let us use the notation of the statement. The last sentence of the statement follows since the first and Thm. 1.4.20. The properties in the statement determine uniquely the maps f_ϕ , T_ϕ and α , let us prove that these properties are well-set and that the remaining of the statement holds. First claim: f_ϕ is well-set and $f_\phi \in \mathbf{Fct}_{\text{top}}([N, V], [M, U])$. Since the naturality of flows, see for instance [18, Prp. 9.13], we deduce that $\phi \circ \vartheta^V(t) = \vartheta^U(t) \circ \phi$, hence

$$\vartheta^U(t)(\phi(Z)) = \phi(\vartheta^V(t)Z) \subseteq \phi(Y);$$

moreover $\phi(Y)$ and $\phi(Z)$ are open since ϕ is open by construction, so $\phi(Y), \phi(Z) \in [M, U]$ and $t \in \text{mor}_{[M, U]}(\phi(Y), \phi(Z))$, hence f_ϕ is well-set. Next f_ϕ on morphisms is clearly continuous and composition preserving, hence our first claim has been proven. Second claim: T_ϕ is well-set and $T_\phi \in \text{Mor}_{\mathbf{Fct}([N, V], \text{tsa})}(F_{[M, U]} \circ f_\phi, F_{[N, V]})$. T_ϕ is well-set since the first proven claim and Prp. 1.6.22. Next $T_\phi(Y)$ clearly is a morphism of $*$ -algebras which in addition is continuous since Def. 1.6.18(3). It remains to show that the following diagram in tsa is commutative

$$\begin{array}{ccc} \mathcal{A}(\phi(Z)) & \xrightarrow{T_\phi(Z)} & \mathcal{A}(Z) \\ \uparrow F_{[M, U]}((\phi(Y), \phi(Z)), t) & & \uparrow F_{[N, V]}((Y, Z), t) \\ \mathcal{A}(\phi(Y)) & \xrightarrow{T_\phi(Y)} & \mathcal{A}(Y) \end{array}$$

Let $h \in \mathcal{A}(\phi(Y))$ then

$$\begin{aligned} (T_\phi(Z) \circ F_{[M, U]}((\phi(Y), \phi(Z)), t))h &= \\ T_\phi(Z)(h \circ \vartheta^U(t) \upharpoonright \phi(Z)) &= \\ h \circ \vartheta^U(t) \circ \phi \upharpoonright Z &= \\ h \circ \phi \circ \vartheta^V(t) \upharpoonright Z &= \\ (h \circ \phi \upharpoonright Y) \circ \vartheta^V(t) \upharpoonright Z &= \\ (F_{[N, V]}((Y, Z), t) \circ T_\phi(Y))h, & \end{aligned}$$

where the 3th equality follows since the naturality of flows, thus the previous diagram is commutative and the second claim is shown. In order to conclude the proof we need to show the following 3th claim: $\alpha(\psi \circ_{\text{vf}_0} \phi) = \alpha(\psi) \circ_{\text{dp}} \alpha(\phi)$ for any $\phi \in \text{Mor}_{\text{vf}_0}((M, U), (N, V))$ and $\psi \in \text{Mor}_{\text{vf}_0}((N, V), (L, W))$. To this end note that $\alpha(\psi \circ_{\text{vf}_0} \phi) = \alpha(\phi \circ \psi) = (f_{\phi \circ \psi}, T_{\phi \circ \psi})$ and

$$\begin{aligned} \alpha(\psi) \circ_{\text{dp}} \alpha(\phi) &= (f_\psi, T_\psi) \circ_{\text{dp}} (f_\phi, T_\phi) \\ &= (f_\phi \circ f_\psi, T_\psi \circ (T_\phi * 1_{f_\psi})) \\ &= (f_\phi \circ f_\psi, T_\psi \circ (T_\phi \circ (f_\psi)_o)), \end{aligned}$$

where the last equality follows since (2), thus our claim is equivalent to the following one

$$\begin{aligned} f_{\phi \circ \psi} &= f_{\phi} \circ f_{\psi}, \\ T_{\phi \circ \psi} &= T_{\psi} \circ (T_{\phi} \circ (f_{\psi})_o). \end{aligned}$$

The first equality is trivially true. Next let $D \in [L, W]$ and $k \in \mathcal{A}((\phi \circ \psi)(D))$ then $T_{\phi \circ \psi}(D)k = (k \circ \phi \upharpoonright \psi(D)) \circ \psi \upharpoonright D$, while

$$\begin{aligned} (T_{\psi} \circ (T_{\phi} \circ (f_{\psi})_o))(D)k &= \\ (T_{\psi}(D) \circ T_{\phi}(\psi(D)))k &= \\ T_{\psi}(D)(k \circ \phi \upharpoonright \psi(D)) &= (k \circ \phi \upharpoonright \psi(D)) \circ \psi \upharpoonright D, \end{aligned}$$

so the second equality and then our 3th claim has been proven. \square

DEFINITION 1.6.25. $\mathbf{a}^n := \Psi \circ \alpha \circ \iota_n$.

COROLLARY 1.6.26 (\mathbf{a}^n is a species). $\mathbf{St}_n \in \mathbf{Cat}$ while $\mathbf{a}^n \in \text{Mor}_{2-\text{dp}}(\mathbf{St}_n, \mathcal{Ch}\mathfrak{d}\mathfrak{v})$, namely $\mathbf{a}^n \in \mathfrak{Sp}(\mathbf{St}_n)$, in particular $1_{\mathbf{a}^n} \in \mathfrak{Set}(\mathbf{a}^n)$.

PROOF. Since Thm. 1.6.24. \square

This result permits the following

DEFINITION 1.6.27. $\mathfrak{u}(\mathbf{a}^n) \equiv$ of classical n -dimensional gravity.

Next we shall see that the equiformity principle $\mathcal{E}(1_{\mathbf{a}^n})(\mathfrak{P}^{\mathbf{a}^n}, G^{\mathbf{a}^n})(1)$ - of the connector, actually a sector, $1_{\mathbf{a}^n}$ evaluated on the standard experimental setting associated with \mathbf{a}^n - is nothing else than the generalization at n dimensions of the equivalence principle in general relativity. In the next section we shall consider instead equiformity principles between \mathbf{a}^n and quantum gravity species.

COROLLARY 1.6.28 (Equiformity principle of $1_{\mathbf{a}^n}$ alias equivalence principle of general relativity). Let $n \in \mathbb{Z}_0^+$, $(\mathcal{M}, U), (\mathcal{N}, V) \in \mathbf{St}_n$, where $\mathcal{M} = (M, g)$ and $\mathcal{N} = (N, h)$. Thus for any $\phi \in \text{Mor}_{\mathbf{St}_n}((\mathcal{M}, U), (\mathcal{N}, V))$, $Y, Z \in [N, V]$ and $t \in \text{mor}_{[N, V]}(Y, Z)$, we have for all $\psi \in \mathfrak{P}_{F_{[N, V]}(Z)}$ and $A \in \mathcal{A}(\phi(Y))_{ob}$

(1) invariant form

$$\mathfrak{f}_{(\psi, A \circ \phi \upharpoonright Y)}^{\mathbf{a}^n, (\mathcal{N}, V), Y, Z}(((Y, Z), t)) = \mathfrak{f}_{(\psi \circ (\phi \upharpoonright Z)^*, A)}^{\mathbf{a}^n, (\mathcal{M}, U), \phi(Y), \phi(Z)}(((\phi(Y), \phi(Z)), t));$$

(2) equivariant form

$$(\phi \upharpoonright Y)^* \mathfrak{t}^{\mathbf{a}^n, (\mathcal{N}, V), Y, Z, \psi} = f_{\phi}^{\top} \mathfrak{t}^{\mathbf{a}^n, (\mathcal{M}, U), \phi(Y), \phi(Z), \psi \circ (\phi \upharpoonright Z)^*}.$$

Moreover if ϕ is a diffeomorphism, then U is geodesic if and only if V is geodesic and if in addition ϕ preserves the orientation then U is a vector field of a perfect fluid on \mathcal{M} if and only if V is a vector field of a perfect fluid on \mathcal{N} .

PROOF. Since Cor. 1.6.26, Thm. 1.4.46(5) and Prp. 1.4.34 applied for the positions $\mathbf{a} = \mathbf{b} = \mathbf{a}^n$ and \mathbf{T} the identity morphism of $\mathbf{Fct}(\mathbf{St}_n, \mathcal{Ch}\mathfrak{d}\mathfrak{v})$ relative to \mathbf{a}^n . The final sentence follows since Thm. 1.6.12. \square

1.6.2. Equiformity principle between classical and quantum gravity. We start the present section by defining the concept of realization which represents the equiformity principle when the context morphism ϕ is trivial. Roughly speaking a species \mathbf{b} realizes a species \mathbf{a} when the dynamics $\tau_{\mathbf{b}}$ mirrors the dynamics $\tau_{\mathbf{a}}$. Since any connector induces an equiformity principle according to Thm. 1.4.46, one shows that the target species of any connector is a realization of its source species Cor. 1.6.30. Based on the structure of \mathbf{a}^n we constructed in Cor. 1.6.26, we define the collection of n -dimensional gravity species, the subcollection of classical ones to which \mathbf{a}^n belongs and diverse subcollections of n -dimensional quantum gravity species. Then we establish that for any connector \mathbf{T} from \mathbf{a}^n to a n -dimensional strict quantum gravity species, exist quantum realizations of the velocity of maximal integral curves of complete vector fields Cor. 1.6.40. As an application to a Robertson-Walker spacetime $M(x, f)$ we establish that the Hubble parameter $H = f'/f$, the acceleration f'' of the scale function and the constraints of its positivity evaluated over a subset of the range of the galactic time t of a geodesic α , are expressed in terms of a quantum realization of the velocity of α Thm. 1.6.41 and Cor. 1.6.43.

DEFINITION 1.6.29 (Realizations). *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{Sp}_*$. We say that \mathbf{b} is a realization of \mathbf{a} or that \mathbf{b} realizes the dynamics of \mathbf{a} if for all $\mathfrak{E}^{\mathbf{b}} = (\mathfrak{S}^{\mathbf{b}}, \mathbf{R}^{\mathbf{b}}) \in \text{Exp}(\mathbf{b})$ there exists $\mathfrak{E}^{\mathbf{a}} = (\mathfrak{S}^{\mathbf{a}}, \mathbf{R}^{\mathbf{a}}) \in \text{Exp}(\mathbf{a})$, for all $M \in d(\mathbf{a})$ and all $y, z \in \mathbf{R}_M^{\mathbf{b}}$ there exist $y', z' \in \mathbf{R}_M^{\mathbf{a}}$, for all $g \in \mathbf{R}_M^{\mathbf{b}}(y, z)$ there exists $g' \in \mathbf{R}_M^{\mathbf{a}}(y', z')$, for all $\psi \in \mathfrak{S}_M^{\mathbf{b}}(z)$ there exists $\psi' \in \mathfrak{S}_M^{\mathbf{a}}(z')$ and for all $A \in \mathcal{A}_{\mathbf{a}(M)}(y')_{ob}$ there exists $A' \in \mathcal{A}_{\mathbf{b}(M)}(y)_{ob}$ satisfying*

$$\mathfrak{f}_{(\psi, A')}^{\mathbf{b}, M, y, z}(g) = \mathfrak{f}_{(\psi', A)}^{\mathbf{a}, M, y', z'}(g').$$

As a result of the equiformity principle the target species of a connector realizes the dynamics of the source species, namely

COROLLARY 1.6.30 (Realizations induced by connectors). *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{Sp}_*$ and $\mathbf{T} \in \mathfrak{Cnt}(\mathbf{a}, \mathbf{b})$, thus \mathbf{b} is a realization of \mathbf{a} such that for all $\mathfrak{E} \in \text{Exp}(\mathbf{b})$ we can select $\mathfrak{E}^{\mathbf{a}} = (\mathfrak{P}^{\mathbf{a}}, \mathbf{G}^{\mathbf{a}})$; while if $\Gamma(\mathfrak{E}, \mathbf{T}) \neq \emptyset^{13}$, then for any $s \in \Gamma(\mathfrak{E}, \mathbf{T})$ we can select $\mathfrak{E}^{\mathbf{a}} = \mathbf{T}[\mathfrak{E}, s]$. Moreover letting $\mathfrak{E} = (\mathfrak{S}, \mathbf{R})$, for any $M \in d(\mathbf{a})$, $y, z \in \mathbf{R}_M$, $g \in \mathbf{R}_M(y, z)$ and $\psi \in \mathfrak{S}_M(z)$ we can select $y' = \mathbf{T}_1^o(M)y$, $z' = \mathbf{T}_1^o(M)z$, $g' = \mathbf{T}_1^m(M)g$ and $\psi' = \mathbf{T}_3^+(M)(z)\psi$, while for any $A \in \mathcal{A}_{\mathbf{a}(M)}(y')_{ob}$ we can select $A' = \mathbf{T}_3(M)(y)A$.*

PROOF. The general case follows since Prp. 1.4.34 applied for $M = N$ and $\phi = 1_M$, and by Thm. 1.4.47(1), the case $\Gamma(\mathfrak{E}, \mathbf{T}) \neq \emptyset$ by Thm. 1.4.46(2). \square

Next we shall define the collection of n -dimensional gravity species as the set of species contextualized on St_n and factorizable through Ψ and ι_n . In this way we provide at the same time a sort of minimal extension of the construction of \mathbf{a}^n , the possibility to select diverse subcollections of quantum gravity species, and most importantly a path to construct connectors between the classical and a quantum gravity.

¹³for instance whenever $\mathbf{T}_1^o(M)$ is injective for all $M \in d(\mathbf{a})$ see Rmk. 1.4.45.

DEFINITION 1.6.31 (n -dimensional gravity species). Let $n \in \mathbb{Z}_0^+$, define

$$\begin{aligned} G_n &:= \{\mathbf{b} \in \mathfrak{Sp}(\mathbf{St}_n) \mid (\exists \mathbf{b} \in \mathbf{Fct}(\mathbf{vf}_0, \mathbf{dp}))(\mathbf{b} = \Psi \circ \mathbf{b} \circ \iota_n)\}, \\ CG_n &:= \{\mathbf{b} \in G_n \mid (\forall M \in \mathbf{St}_n)(\forall x \in G_M^b)(\mathcal{A}_{\mathbf{b}(M)}(x) \text{ is commutative})\}; \\ QG_n &:= \{\mathbf{b} \in G_n \mid (\forall M \in \mathbf{St}_n)(\forall x \in G_M^b)(\mathcal{A}_{\mathbf{b}(M)}(x) \text{ is noncommutative})\}; \\ sQG_n &:= \{\mathbf{b} \in QG_n \mid (\forall M \in \mathbf{St}_n)(G_M^b = G_M^{a^n})\}. \end{aligned}$$

We call

- (1) G_n the collection of n -dimensional gravity species;
- (2) CG_n the collection of n -dimensional classical gravity species;
- (3) QG_n the collection of n -dimensional quantum gravity species;
- (4) sQG_n the collection of n -dimensional strict quantum gravity species.

Clearly $\mathbf{a}^n \in CG_n$. We can define the relativistic species roughly speaking by restriction of 4-dimensional gravity species \mathbf{b} over the category of vector fields on subregions of a fixed Minkowski spacetime, namely \mathbf{b} composed the identity functor valued in \mathbf{St}_4 and defined in the full subcategory $\mathbf{Sr}(\mathbb{M})$ of submanifolds of a given Minkowski spacetime \mathbb{M} and observer fields on them.

DEFINITION 1.6.32. Let \mathbb{M} be a fixed Minkowski spacetime. Define $\mathbf{Sr}(\mathbb{M})$ to be the full subcategory of \mathbf{St}_4 whose object set is the subset of all the (\mathcal{M}, U) such that \mathcal{M} is a semi-Riemannian submanifold of \mathbb{M} . Moreover let $\mathbf{sr}(\mathbb{M})$ be the subcategory of $\mathbf{Sr}(\mathbb{M})$ whose object set is $\mathbf{Sr}(\mathbb{M})$ and for any object P, Q we have that $\mathbf{Mor}_{\mathbf{sr}(\mathbb{M})}(P, Q)$ is the subset of those elements of $\mathbf{Mor}_{\mathbf{Sr}(\mathbb{M})}(P, Q)$ which are local diffeomorphisms.

Note that if for any $P \in \mathbf{sr}(\mathbb{M})$ and $p \in P$, we identify the tangent space $T_p P$ with the Minkowski 4-space \mathbb{R}_1^4 , then $\{d\phi_p \mid \phi \in \mathbf{Mor}_{\mathbf{sr}(\mathbb{M})}(P, P)\} = O_1(4)$ the subgroup of linear isometries of the space \mathbb{R}_1^4 . We recall that given a category B and a subcategory A of B , thus $\mathbf{l}_{A \rightarrow B}$ denotes the functor from A to B whose object and morphism maps are the identity maps.

DEFINITION 1.6.33. Define $\mathbf{l}_{\mathbb{M}} := \mathbf{l}_{\mathbf{Sr}(\mathbb{M}) \rightarrow \mathbf{St}_4}$, $\iota^{\mathbb{M}} := \iota_4 \circ \mathbf{l}_{\mathbb{M}}$, and $\mathbf{a}^{rt} := \mathbf{a}^4 \circ \mathbf{l}_{\mathbb{M}}$

DEFINITION 1.6.34. Define

$$\begin{aligned} RT(\mathbb{M}) &:= \{\mathbf{c} \in \mathfrak{Sp}(\mathbf{Sr}(\mathbb{M})) \mid (\exists \mathbf{b} \in G_4)(\mathbf{c} = \mathbf{b} \circ \mathbf{l}_{\mathbb{M}})\}; \\ CRT(\mathbb{M}) &:= \{\mathbf{c} \in \mathfrak{Sp}(\mathbf{Sr}(\mathbb{M})) \mid (\exists \mathbf{b} \in CG_4)(\mathbf{c} = \mathbf{b} \circ \mathbf{l}_{\mathbb{M}})\}; \\ QRT(\mathbb{M}) &:= \{\mathbf{c} \in \mathfrak{Sp}(\mathbf{Sr}(\mathbb{M})) \mid (\exists \mathbf{b} \in QG_4)(\mathbf{c} = \mathbf{b} \circ \mathbf{l}_{\mathbb{M}})\}; \\ sQRT(\mathbb{M}) &:= \{\mathbf{c} \in \mathfrak{Sp}(\mathbf{Sr}(\mathbb{M})) \mid (\exists \mathbf{b} \in sQG_4)(\mathbf{c} = \mathbf{b} \circ \mathbf{l}_{\mathbb{M}})\}. \end{aligned}$$

We call

- (1) $RT(\mathbb{M})$ the collection of relativistic species in \mathbb{M} ;
- (2) $CRT(\mathbb{M})$ the collection of classical relativistic species in \mathbb{M} ;
- (3) $QRT(\mathbb{M})$ the collection of quantum relativistic species in \mathbb{M} ;
- (4) $sQRT(\mathbb{M})$ the collection of strict quantum relativistic species in \mathbb{M} .

REMARK 1.6.35. Clearly $\mathbf{a}^{rt} \in CRT(\mathbb{M})$ moreover we have that

$$\begin{aligned} RT(\mathbb{M}) &\subseteq \{\mathbf{c} \in \mathfrak{Sp}(\mathbf{Sr}(\mathbb{M})) \mid (\exists \mathbf{b} \in \mathbf{Fct}(\mathbf{vf}_0, \mathbf{dp}))(\mathbf{c} = \Psi \circ \mathbf{b} \circ \iota^{\mathbb{M}})\}; \\ CRT(\mathbb{M}) &\subseteq \{\mathbf{c} \in RT(\mathbb{M}) \mid (\forall P \in \mathbf{Sr}(\mathbb{M}))(\forall x \in G_p^c)(\mathcal{A}_{c(P)}(x) \text{ is commutative})\}; \\ QRT(\mathbb{M}) &\subseteq \{\mathbf{c} \in RT(\mathbb{M}) \mid (\forall P \in \mathbf{Sr}(\mathbb{M}))(\forall x \in G_p^c)(\mathcal{A}_{c(P)}(x) \text{ is noncommutative})\}; \\ sQRT(\mathbb{M}) &\subseteq \{\mathbf{c} \in QRT(\mathbb{M}) \mid (\forall P \in \mathbf{Sr}(\mathbb{M}))(G_p^c = G_p^{\mathbf{a}^{rt}})\}. \end{aligned}$$

Next let us come back to the general case and examine the outcomes of a realization of \mathbf{a}^n belonging to sQG_n . For any context (\mathcal{M}, U) in \mathbf{St}_n and any open set Z of \mathcal{M} , there exists a variety of observables in $\mathcal{A}(Z)$ obtained by using the metric tensor of \mathcal{M} and the vector field U , indeed $\langle B, U \rangle \upharpoonright Z \in \mathcal{A}(Z)_{ob}$ for any smooth vector field B on \mathcal{M} in particular any component of a frame field on \mathcal{M} . Thus let us set

DEFINITION 1.6.36. Let $n \in \mathbb{Z}_0^+$, $O = (\mathcal{M}, U) \in \mathbf{St}_n$, where $\mathcal{M} = (M, g_{\mathcal{M}})$, $\mathcal{F} = \{E_r\}_{r=1}^n$ be a frame field of \mathcal{M} , $Y, Z \in Op(M)$, $t \in \mathbb{R}$ satisfying $\theta^U(t)Z \subseteq Y$, $\chi \in \mathfrak{P}_{F_{[\mathcal{M}, U]}(Z)}$ thus for any $k \in \{1, \dots, n\}$ we define

$$u_{k, \chi, \mathcal{F}}^{O, Y, Z}(t) := \chi(\langle E_k, U \rangle_{\mathcal{M}} \circ \theta^U(t) \upharpoonright Z).$$

If in addition the topology $\xi_{[\mathcal{M}, U]}$ is stronger than or equal to the pointwise topology on $\mathcal{A}(M)$, then for all $p \in Z$ set $u_{k, p, \mathcal{F}}^{O, Y, Z} := u_{k, \delta_p^Z, \mathcal{F}}^{O, Y, Z}$, where δ_p^Z is the Dirac distribution on $\mathcal{A}(Z)$ centered in p .

REMARK 1.6.37. If the topology $\xi_{[\mathcal{M}, U]}$ is stronger than the pointwise topology on $\mathcal{A}(M)$, then the topological dual of $\mathcal{A}(Z)$ w.r.t. the pointwise topology is a subset of the topological dual of $F_{[\mathcal{M}, U]}(Z)$, hence $u_{k, p, \mathcal{F}}^{O, Y, Z}$ is well defined. Moreover since for all $p \in Z$ and $t \in \mathbb{R}$ we have $\theta^U(t)p = \alpha_p^U(t)$ and $U(\alpha_p^U(t)) = (\alpha_p^U)'(t)$, we obtain if $\theta^U(t)Z \subseteq Y$

$$u_{k, p, \mathcal{F}}^{O, Y, Z}(t) = g_{\mathcal{M}}(\alpha_p^U(t)) (E_k(\alpha_p^U(t)), (\alpha_p^U)'(t)),$$

where we recall that α_p^U is the maximal integral curve of U such that $\alpha_p^U(0) = p$, while $(\alpha_p^U)'(t)$ is the velocity vector of α_p^U at t .

CONVENTION 1.6.38. Let $n \in \mathbb{Z}_0^+$, $\mathbf{b} \in sQG_n$, $O = (\mathcal{M}, U) \in \mathbf{St}_n$, where $\mathcal{M} = (M, g_{\mathcal{M}})$, $Y, Z \in Op(M)$ and $t \in mor_{[\mathcal{M}, U]}(Y, Z)$. Then let $\tau_{\mathbf{b}(O)}(t)$ denote $\tau_{\mathbf{b}(O)}(((Y, Z), t))$ whenever it will not cause confusion.

Next we will apply Cor. 1.6.30 when \mathbf{a} is the classical n -dimensional gravity and \mathbf{b} is a n -dimensional strict quantum gravity, but let us start with the following

DEFINITION 1.6.39. Let $n \in \mathbb{Z}_0^+$, $\mathbf{b} \in sQG_n$, $O = (\mathcal{M}, U) \in \mathbf{St}_n$, where $\mathcal{M} = (M, g_{\mathcal{M}})$, $\mathcal{F} = \{E_r\}_{r=1}^n$ be a frame field of \mathcal{M} and $Y, Z \in Op(M)$. We define $\{U_r\}_{r=1}^n$ to be a quantum realization from Y to Z through L of the velocity of the maximal integral curves of U on \mathcal{M} and relative to \mathcal{F} , if $\{U_r\}_{r=1}^n \subset \mathcal{A}_{\mathbf{b}(O)}(Y)_{ob}$ and if there exist $Y', Z' \in Op(M)$ with the following properties. $L : mor_{[\mathcal{M}, U]}(Y, Z) \rightarrow mor_{[\mathcal{M}, U]}(Y', Z')$ and for any $\psi \in \mathfrak{P}_{\mathcal{A}_{\mathbf{b}(O)}(Z)}$ there exists $\chi \in \mathfrak{P}_{F_{[\mathcal{M}, U]}(Z')}$ such that for all $t \in mor_{[\mathcal{M}, U]}(Y, Z)$ and all $k \in \{1, \dots, n\}$ we have

$$\psi(\tau_{\mathbf{b}(O)}(t)U_k) = u_{k, \chi, \mathcal{F}}^{O, Y', Z'}(L(t)).$$

We call the map $t \mapsto \psi(\tau_{b(O)}(t)U_k)$ on $\text{mor}_{[M,U]}(Y, Z)$ the k -evaluation of $\{U_r\}_{r=1}^n$ in ψ . Now we can state the following

COROLLARY 1.6.40 (Quantum realization of the velocity of maximal integral curves of complete vector fields). *Let $n \in \mathbb{Z}_0^+$, $\mathbf{b} \in sQG_n$, $\mathbf{T} \in \mathfrak{C}nt(\mathbf{a}^n, \mathbf{b})$ and $O = (\mathcal{M}, U) \in \mathbf{St}_n$, where $\mathcal{M} = (M, g_{\mathcal{M}})$. Thus for any $Y, Z \in \text{Op}(M)$ and any frame field $\mathcal{F} = \{E_r\}_{r=1}^n$ of \mathcal{M} there exists a quantum realization $\{U_r\}_{r=1}^n$ from Y to Z through $\mathbf{T}_1^m(O)$ of the velocity of the maximal integral curves of U on \mathcal{M} and relative to \mathcal{F} . Namely there exist $Y', Z' \in \text{Op}(M)$ with the following properties. For any $\psi \in \mathfrak{P}_{\mathcal{A}_{b(O)}(Z)}$ there exists $\chi \in \mathfrak{P}_{F_{[M,U]}(Z')}$ such that for all $t \in \mathbb{R}$ satisfying $\theta^U(t)Z \subseteq Y$ and all $k \in \{1, \dots, n\}$ we obtain*

$$(30) \quad \begin{aligned} \psi(\tau_{b(O)}(t)U_k) &= u_{k,\chi,\mathcal{F}}^{O,Y',Z'}(\bar{t}), \\ \bar{t} &= \mathbf{T}_1^m(O)(t). \end{aligned}$$

Moreover we can select $Y' = \mathbf{T}_1^o(O)Y$, $Z' = \mathbf{T}_1^o(O)Z$, $\chi = \mathbf{T}_3^+(O)(Z)\psi$, and $U_k = \mathbf{T}_3(O)(Y)A$ with $A = \langle E_k, U \rangle_{\mathcal{M}} \upharpoonright Y'$ for any $k \in \{1, \dots, n\}$. If in addition the topology $\xi_{[M,U]}$ is stronger than or equal to the pointwise topology on $\mathcal{A}(M)$ and if there exists $p \in Z'$ such that $\chi = \delta_p^{Z'}$, then

$$(31) \quad \psi(\tau_{b(O)}(t)U_k) = g_{\mathcal{M}}(\alpha_p^U(\bar{t})) \left(E_k(\alpha_p^U(\bar{t})), (\alpha_p^U)'(\bar{t}) \right).$$

PROOF. According to Cor. 1.6.30 \mathbf{b} is a realization of \mathbf{a}^n such that we can select $\mathfrak{E}^b = (\mathfrak{P}^b, G^b)$, $\mathfrak{E}^{a^n} = (\mathfrak{P}^{a^n}, G^{a^n})$, Y', Z', χ as in the statement and $A' = \mathbf{T}_3(O)(Y)A$. Thus (30) follows since Def. 1.6.29 applied to the object O and to $A = \langle E_k, U \rangle_{\mathcal{M}} \upharpoonright Y' \in F_{[M,U]}(Y')_{ob}$. (31) follows since (30) and Rmk. 1.6.37. \square

Let $M(x, f) = I \times_f S$ be the Robertson-Walker spacetime with sign x and scale function f see [25, Def. 12.7]. Here concerning $M(x, f)$ we follow the general notation in [25, p. 204] in particular π (galactic time) and σ are the projections defined on $M(x, f)$ onto I and S respectively. Let $H : I \rightarrow \mathbb{R}$ be the Hubble parameter relative to $M(x, f)$ defined $H := f'/f$, where f' is the derivative of the map f , see for instance [24, eq (1.10)], see also [25, p.347]. Let H' denote the derivative of the map H .

Now let \mathbf{T} be a connector from \mathbf{a}^4 to a 4-dimensional strict quantum gravity species and α be a geodesic in $M(x, f)$ such that it is complete one of the existing vector fields V of $M(x, f)$ for which a possibly restriction of α is an integral curve. Moreover let $\{E_r\}_{r=1}^3$ be a frame field on S , $t = \pi \circ \alpha$ be the galactic time of α and t_0 be a suitable restriction of $t \circ \mathbf{T}_1^m(O)$ where $O = (M(x, f), V)$. In Thm. 1.6.41 (see also (37) in a more compact form) we establish that $H \circ t_0$ is the quotient of the classical factor c_k and the quantum factor q_k equal to the k -evaluation of a quantum realization of the velocity of α relative to any frame field on $M(x, f)$ extending the lift of $\{E_r\}_{r=1}^3$ to $M(x, f)$.

Since [25, Prp. 12.22(2)] $H \circ t$ enters in the expression of the velocity β' of the projection $\beta = \sigma \circ \alpha$. Thus the strategy is to extract the component β' from the velocity α' by using the lift to $M(x, f)$ of $\{E_r\}_{r=1}^3$ and then to apply Cor. 1.6.40 to any frame field on $M(x, f)$ extending this lift and to the complete vector field V .

THEOREM 1.6.41 (Quantum and classical factors of the Hubble parameter). *Let $\mathbf{b} \in sQG_4$, $\mathbf{T} \in \mathfrak{C}nt(\mathfrak{a}^4, \mathbf{b})$ and α be a geodesic on $M(x, f)$. Assume that $0 \in \text{dom}(\alpha)$ eventually by a reparametrization, that α is regular in 0, that it is complete one of the vector fields V on $M(x, f)$ for which the restriction of α to an open neighbourhood K_0 of 0 is an integral curve of V . Thus for any $Y, Z \in \text{Op}(M(x, f))$ and any frame field $\{E_r\}_{r=1}^3$ of S , said $O = (M(x, f), V)$ there exists $\{V_r\}_{r=1}^3 \subset \mathcal{A}_{\mathbf{b}(O)}(Y)_{ob}$ with the following properties. For any $\psi \in \mathfrak{P}_{\mathcal{A}_{\mathbf{b}(O)}(Z)}$, for all $s \in \mathbb{R}$ satisfying $\theta^V(s)Z \subseteq Y$ and for all $k \in \{1, 2, 3\}$ we obtain*

$$(32) \quad \begin{aligned} \psi(\tau_{\mathbf{b}(O)}(s)V_k) &= \chi(\langle \bar{E}_k, V \rangle_{M(x, f)} \circ \theta^V(\bar{s}) \upharpoonright Z'), \\ \bar{s} &:= \mathbf{T}_1^m(O)(s), \end{aligned}$$

where $\chi = \mathbf{T}_3^+(O)(Z)\psi$, \bar{E}_k is the lift of E_k to $M(x, f)$ and $Z' = \mathbf{T}_1^o(O)Z$. Moreover said $Y' = \mathbf{T}_1^o(O)Y$ we can select $V_k = \mathbf{T}_3(O)(Y)(\langle \bar{E}_k, V \rangle_{M(x, f)} \upharpoonright Y')$. Set

$$K^{Y, Z} := \{\lambda \in \mathbb{R} \mid \theta^V(\lambda)Z \subseteq Y, \mathbf{T}_1^m(O)(\lambda) \in K_0\}.$$

If the topology $\xi_{[M(x, f), V]}$ is stronger than or equal to the pointwise topology on $\mathcal{A}(M(x, f))$, $\alpha(0) \in Z'$ and $\chi = \delta_{\alpha(0)}^{Z'}$ then for all $s \in K^{Y, Z}$ we obtain

$$(33) \quad H(t(\bar{s})) = \frac{-f^2(t(\bar{s}))g_S(\beta(\bar{s}))(E_k(\beta(\bar{s})), \beta''(\bar{s}))}{2\psi(\tau_{\mathbf{b}(O)}(s)V_k)} \left(\frac{dt}{ds}(\bar{s}) \right)^{-1},$$

where $t = \pi \circ \alpha$ and $\beta = \sigma \circ \alpha$.

PROOF. Let α be a geodesic on $M(x, f)$ such that $0 \in \text{dom}(\alpha)$ eventually by a reparametrization, and let $t = \pi \circ \alpha$ and $\beta = \sigma \circ \alpha$. Assume α be regular in 0 then by an application of [25, Ex. 3.12] we can find local vector fields N of I and J of S , and an open neighbourhood K_0 of 0 in $\text{dom}(\alpha)$ such that N is defined on an open neighbourhood of $t(0)$, J is defined on an open neighbourhood $\beta(0)$ and $t \upharpoonright K_0$ and $\beta \upharpoonright K_0$ are integral curves of N and J respectively. Now since [25, Ex. 1.18] we deduce that N and J admit extensions to vector fields on I and S respectively, let us denote such extensions with the same symbols, thus we can take the lifts \bar{N} and \tilde{J} on $M(x, f)$ of N and J respectively. It is easy to see that $\alpha \upharpoonright K_0$ is an integral curve of $V = \bar{N} + \tilde{J}$, moreover for any $r \in K_0$ we have that $\alpha'(r) = \tilde{t}'(r) + \tilde{\beta}'(r)$, where $\tilde{t}'(r)$ and $\tilde{\beta}'(r)$ are the lifts on $M(x, f)$ of $t'(r)$ and $\beta'(r)$ respectively. We recall that g_N and $\langle \cdot, \cdot \rangle_N$ denote the metric tensor of any semi-Riemannian manifold N , thus by the above equality, for any frame field $\{E_k\}_{k=1}^3$ of S and by letting \bar{E}_k be the lift of E_k on $M(x, f)$ we obtain for all $r \in K_0$

$$(34) \quad g_{M(x, f)}(\alpha(r))(\bar{E}_k(\alpha(r)), \alpha'(r)) = f^2(t(r))g_S(\beta(r))(E_k(\beta(r)), \beta'(r)).$$

Now if V is complete, then we can apply Cor. 1.6.40 to $O = (M(x, f), V)$ and to any frame field on $M(x, f)$ extending $\{\bar{E}_k\}_{k=1}^3$, thus (32) follows since (30). Next $\alpha \upharpoonright K_0$ is an integral curve of V , thus since the uniqueness of the integral curves see for instance [25, Cor. 1.50], we obtain by (34) that for all $r \in K_0$ and $k \in \{1, 2, 3\}$

$$(35) \quad (\langle \bar{E}_k, V \rangle_{M(x, f)} \circ \theta^V(r))(\alpha(0)) = f^2(t(r))g_S(\beta(r))(E_k(\beta(r)), \beta'(r)).$$

Now we can use [25, Prp. 12.22(2)] to express β' in terms of the Hubble parameter to obtain by (35)

$$(36) \quad (\langle \bar{E}_k, V \rangle_{M(x,f)} \circ \theta^V(r))(\alpha(0)) = \frac{-f^2(t(r))g_s(\beta(r))(E_k(\beta(r)), \beta''(r))}{2H(t(r))} \left(\frac{dt}{ds}(r) \right)^{-1}.$$

(33) follows since (36) and (32). \square

DEFINITION 1.6.42. Assume the hypothesis and notation of Thm. 1.6.41. For all $k \in \{1, 2, 3\}$ define

$$\begin{aligned} q_k &: K^{Y,Z} \rightarrow \mathbb{R} \quad s \mapsto \psi(\tau_{b(O)}(s)V_k), \\ e_k &: K_0 \rightarrow \mathbb{R} \quad r \mapsto -\frac{1}{2}f^2(t(r))g_s(\beta(r))(E_k(\beta(r)), \beta''(r)) \left(\frac{dt}{ds}(r) \right)^{-1}; \\ c_k &:= e_k \circ T_1^m(O) \upharpoonright K^{Y,Z}, \\ t_0 &:= t \circ T_1^m(O) \upharpoonright K^{Y,Z}, \\ q_k &:= q_k \upharpoonright \mathring{K}^{Y,Z}, \\ c_k &:= c_k \upharpoonright \mathring{K}^{Y,Z}, \\ t_0 &:= t_0 \upharpoonright \mathring{K}^{Y,Z}; \end{aligned}$$

with $\mathring{K}^{Y,Z}$ the interior of $K^{Y,Z}$. Call q_k and c_k as well their restrictions to $\mathring{K}^{Y,Z}$ the quantum and classical factors of the Hubble parameter relative to T , $\{E_r\}_{r=1}^3$, Y, Z and k respectively. Let q'_k, c'_k and dt_0/ds be the derivatives of the maps q_k, c_k and t_0 respectively. Define

$$c_k^\pm := \frac{1}{2} \left(\frac{dt_0}{ds} \right)^{-1} \left(q'_k \pm \left((q'_k)^2 - 4c'_k q_k \frac{dt_0}{ds} \right)^{1/2} \right).$$

ans set for all $s \in \mathring{K}^{Y,Z}$

$$\mathfrak{Z}(T, O, k, s) :=] - \infty, c_k^-(s)[\cup] c_k^+(s) + \infty[.$$

The above designations emerge since q_k enrols only the quantum system $b(O)$, c_k engages only the classical system $a^4(O)$ and since according to (33) we have for any $k \in \{1, 2, 3\}$

$$(37) \quad H \circ t_0 = \frac{c_k}{q_k}.$$

By using the above equality, in the next result we express $(f''/f) \circ t_0$ and the conditions for the positivity of $f'' \circ t_0$ as functions of q_k, c_k and their derivatives.

COROLLARY 1.6.43 (Acceleration in terms of the quantum and classical factors of the Hubble parameter). Assume the hypothesis and notation of Thm. 1.6.41. Thus for all $k \in \{1, 2, 3\}$ we have

$$(38) \quad \frac{f''}{f} \circ t_0 = q_k^{-2} \left(c_k^2 - (c_k q'_k - c'_k q_k) \left(\frac{dt_0}{ds} \right)^{-1} \right).$$

In particular for any $s \in \mathring{K}^{YZ}$ if $c_k^\pm(s) \in \mathbb{C} - \mathbb{R}$ then $f''(t_0(s)) > 0$, while if $c_k^\pm(s) \in \mathbb{R}$, then

$$(39) \quad f''(t_0(s)) > 0 \Leftrightarrow c_k(s) \in \mathfrak{I}(\mathsf{T}, O, k, s).$$

PROOF. Let $k \in \{1, 2, 3\}$ thus $f''/f = H' + H^2$ on I , clearly $(H' \circ t_0) = (H \circ t_0)'(\frac{dt_0}{ds})^{-1}$. Thus (38) and (39) follow since (37). \square

Let us discuss the results obtained in the present section. We define the concept of realization as the equiformity principle applied for identity context morphisms Def. 1.6.29. Then the target species of a connector realizes the dynamics of the source species as a result of its equiformity principle Cor. 1.6.30. We define the collection of gravity species and subcollections of quantum gravity species modeled by the structure of \mathfrak{a}^n Def. 1.6.31. Thus we apply Cor. 1.6.30 to establish the existence of quantum realizations of the velocity of the maximal integral curves of complete vector fields on spacetimes (Def. 1.6.39) provided the existence of a connector from \mathfrak{a}^n to an n -dimensional strict quantum gravity Cor. 1.6.40.

Then we employ this result in the special case of a Robertson-Walker spacetime $M(x, f)$ with sign k and scale function f . The outcomes can be so summarized. Let T be a connector from \mathfrak{a}^4 to a 4-dimensional strict quantum gravity species \mathfrak{b} and α be a geodesic on $M(x, f)$, then under the hypothesis of Thm. 1.6.41 we establish what follows.

- (1) There exists a quantum realization of the velocity of α Thm. 1.6.41(32).
- (2) $H \circ t_0 = c_k/q_k$ Thm. 1.6.41(33) (see (37)). Here H is the Hubble parameter, t_0 is the restriction on K^{YZ} of the composition of the galactic time of α with $\mathsf{T}_1^m(O)$, while q_k and c_k are the quantum and classical factors of the Hubble parameter Def. 1.6.42.
- (3) $(f''/f) \circ t_0$ is a function of q_k, c_k and their derivatives Cor. 1.6.43(38), where q_k, c_k and t_0 are the restrictions of q_k, c_k and t_0 to the interior \mathring{K}^{YZ} of K^{YZ} respectively.
- (4) For any $s \in \mathring{K}^{YZ}$ we have that $f''(t_0(s)) > 0$ is equivalent to $c_k(s) \in \mathfrak{I}(\mathsf{T}, O, k, s)$ constraining the quantum and classical factors of the Hubble parameter Cor. 1.6.43(39).

It is well-known that for a Robertson-Walker perfect fluid with energy density ρ and pressure p see for example [25, Cor. 12.12], the detected positivity of the acceleration f'' [27, 26] is equivalent to the negative pressure $p < -\rho/3$, occurrence commonly ascribed to the dark energy [24, p. 65].

However for any $s \in \mathring{K}^{YZ}$, $f''(t(\bar{s})) > 0$ namely the positivity of the acceleration when evaluated over the galactic time $t(\bar{s}) = t_0(s)$ of $\alpha(\bar{s})$, as well the negative pressure $p(\alpha(\bar{s})) < -\rho(\alpha(\bar{s}))/3$, are explained via the equivalence (39) as a consequence of the existence of a connector T , from \mathfrak{a}^4 to a strict quantum gravity species, satisfying the constraints $c_k(s) \in \mathfrak{I}(\mathsf{T}, O, k, s)$, rather than the existence of dark energy.

Diagrammatically we can summarize what said as follows. If we assume the hypothesis of Thm. 1.6.41 and let $\mathfrak{D}\mathfrak{e}$ denote the dark energy hypothesis, then for all $s \in \mathring{K}^{YZ}$

and $k \in \{1, 2, 3\}$ we have

$$\begin{array}{ccc}
 \boxed{\begin{array}{l} \mathsf{T} \in \mathfrak{C}\mathfrak{nt}(\mathbf{a}^4, \mathbf{b}) \\ c_k(s) \in \mathfrak{I}(\mathsf{T}, O, k, s) \end{array}} & \xRightarrow{(39)} & \boxed{f''(t(\bar{s})) > 0} \\
 & & \updownarrow \\
 \boxed{\mathfrak{De}} & \xRightarrow{\quad} & \boxed{p(\alpha(\bar{s})) < -\frac{\rho(\alpha(\bar{s}))}{3}}
 \end{array}$$

It is aim of a future work to find $\mathsf{T} \in \mathfrak{C}\mathfrak{nt}(\mathbf{a}^4, \mathbf{b})$ such that $c_k(s) \in \mathfrak{I}(\mathsf{T}, O, k, s)$. We conclude with the following observation. If any of the following occurrences

- (1) (37) by replacing $H \circ t_0$ with the constant map equal to the Hubble parameter at the present time $73.02 \pm 1.79 \text{ km s}^{-1} \text{ Mpc}^{-1}$ as determined by Riess et al. in [28],
- (2) $c_k(s) \in \mathfrak{I}(\mathsf{T}, O, k, s)$, for any $s \in \mathring{K}^{YZ}$;

would be experimentally confirmed, then the empirical existence of T as stated in Posit 1.5.16(4a), in particular that the equiformity principle of T is part of its empirical representation, will be experimentally validated.

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